Robust Recovery Risk Hedging: Only the First Moment Matters*

by

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Credit derivatives are subject to at least two sources of risk: the default time and the recovery payment. This paper examines the impact of modeling the recovery payment on hedging strategies in reduced-form models. We show that all hedging approaches based on a quadratic criterion do only depend on the *expected* recovery payment at default and not the whole shape of the recovery payment distribution if the underlying hedging instrument (say, a defaultable zero coupon bond with total loss in case of default, or common stock) jumps to/or reaches a pre-specified value when the credit event occurs. This justifies assuming a *certain* recovery rate conditional on default time and interest rate level. Hence, this result allows a simplified modeling of credit risk. Moreover, in contrast to the existing literature, our model yields explicit solutions for the hedge ratio even when all relevant quantities are stochastic.

JEL: C10, G13, G24

Key words: Credit Risk, Recovery Risk, Hedging

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Contents

1 Introduction

In contrast to a large amount of theoretical and empirical work available on the valuation of credit derivatives (see Bielecki and Rutkowski (2002), Duffie and Singleton (2003), Lando (2004) for reviews), hedging of credit derivatives remains a largely unexplored avenue of research. When valuing and hedging credit derivatives, two quantities are crucial. The first is the *probability of default* (or default intensity, if it exists), and the second is the *default recovery* (or recovery rate) in the event of default. While in traditional models the recovery rate is given exogenously as a known constant at the default time[1], this rate is stochastic in reality, even conditional on the default time. This uncertainty in the *default recoveries* of both the underlying instrument (e.g., equity) and particularly the credit derivative (e.g., a convertible bond) is perhaps the most important reason why hedges in practice are not self-financing.

The main purpose of this paper is therefore not valuation but hedging credit derivatives in the presence of recovery risk in a reduced-form framework. Since in general, the common objective of arbitrageurs in credit derivatives markets is to minimize the variance of the hedging costs, we focus on the *locally risk-minimizing* hedging strategy. Föllmer and Sondermann (1986) pioneered this approach in the special case where the underlying instrument follows a martingale. At each point in time they require that the risk, defined as the expected quadratic hedging costs, is minimized. However, in semimartingale models a risk-minimizing strategy does not always exist. Therefore, Schweizer (1991) introduced a locally risk-minimizing (LRM) hedging strategy and showed that – under certain assumptions – a strategy is *locally* risk-minimizing if the cost process is a martingale which is orthogonal to the martingale part of the underlying instrument process. The LRM-strategy is *mean-self-financing*, that is at each point in time the expected sum of discounted cash infusions or withdrawals until maturity is zero. The value of the hedge portfolio is then the discounted expected terminal payoff of the option under the so-called *minimal equivalent martingale measure*.

Hedging strategies for credit derivatives within the reduced-form framework have been studied in the literature. On the one hand, there exist quite tractable models where the hedge ratio is explicitly given. For instance, Bielecki, Jeanblanc and Rutkowski (2007) derived a hedging strategy for credit derivatives using credit default swaps (CDS) and a position in the riskless money market account. The model

[1] One exception is Guo, Jarrow and Zeng (2009). They model the recovery rate process itself.

is easily implemented due to the fact that the interest rate level is assumed to be flat at level null and both the default intensity and the recovery payment are deterministic, i.e. the default time is the only random quantity. On the other hand, there exist models that allow all of the relevant quantities to be stochastic, but only yield hedge ratios that contain the predictable process appearing in the above mentioned martingale representation of the claim to hedge. Therefore, using these strategies, one has to calculate this process. If all relevant quantities are stochastic and possibly dependent, the situation quickly becomes hopeless. Models of this type can be found, for instance, in Bielecki, Jeanblanc and Rutkowski (2008) and Bielecki, Jeanblanc and Rutkowski (2011).[2] Biagini and Cretarola (2007, 2009, 2012) applied the local risk-minimization approach to credit derivatives. However, they assume the recovery payment to be *constant conditional on default*, and explicit solutions are given only for the case of either the interest rate or the default rate being stochastic. In this paper, we try to fill the gap between those two classes of models and derive the locally risk-minimizing hedging strategy in the case that the recovery payment is *stochastic conditional on default* and both stochastic but independent interest and default rates. This independence assumption, however, will turn out to be no major restriction.

We derive LRM-hedging strategies for reduced-form models when there are two hedging instruments: a locally riskless money market account and a risky underlying instrument. We denote the recovery rate as *single-stochastic* if the recovery amount depends only on the default event and the interest rate. We call the recovery rate *doubly-stochastic* if the recovery amount also depends on the realization of another random variable. Corresponding model variants are examined for the reduced-form model framework. In this framework we assume the existence of a tradable zero coupon bond with total loss at default of the firm under consideration. However, we emphasize that the defaultable zero coupon bond can be replaced by stocks, if the stock is assumed to fall to a prespecified level at the time of default.

It turns out that the corresponding LRM-strategy is not only mean-self-financing but also self-financing if the default recovery is single-stochastic. That is, as long as the recovery amount is known in the event of default, there exists a self-financing replication strategy for credit derivatives. Moreover, we find that in the more realistic case of doubly-stochastic default recoveries, the LRM-hedging strategy does

[2] In fact, there exists a large amount of sometimes overlapping published and unpublished papers by the same and related authors. For a complete list, we refer to Bielecki and Rutkowski (2002) and Chesney, Jeanblanc and Yor (2009).

only depend on the *expected* recovery amount, not on other characteristics of its distribution. This key result of the paper helps to justify the frequently made simplifying assumption that the default recovery is a constant, conditional on the default event, when valuing and hedging credit derivatives.

At first glance this result seems to contradict the result of Grünewald and Trautmann (1996) when deriving LRM-strategies for stock options in the presence of jump risk. In that setting the LRM-strategy depends additionally on the variance of the stock's jump amplitude. This key difference is due to the fact that in our model default of the firm implies that the underlying instrument's price jumps always to *zero* while in Merton's (1976) jump diffusion setting assumed by Grünewald and Trautmann (1996), the option's underlying stock price jumps to an arbitrary price level.

We also run a simulation to test the impact of the different model assumptions on the cumulative hedging costs. It will turn out that the latter are nearly unaffected by the whether the interest rate is deterministic or stochastic. However, they are affected by the assumptions imposed on the default rate. Therefore, our simulation results suggest that both the recovery and the default rate should be modelled as stochastic processes when hedging credit derivatives. We also test the LRM-strategy against alternative strategies (and alternative hedging instruments). First, we consider the duplication strategy using CDS contracts by Bielecki, Jeanblanc and Rutkowski (2007) or Bielecki, Jeanblanc and Rutkowski (2008), respectively. Finally, we also consider two cross-hedging strategies. The first of them involves a hedging instrument that trades at a spread (in the default intensity) relative to the credit derivative we wish to hedge. The second cross-hedging strategy involves a position in a credit index of the type investigated in Brigo and Morini (2011), i.e. a pool of credit names with the same credit quality (the same default rate) as the instrument we wish to hedge.

The paper is organized as follows: Section 2 describes hedging as a sequential regression and illustrates the paper's basic insight. Section 3 looks at locally risk-minimizing hedging policies in a reduced-form model when recovery is single-stochastic and doubly-stochastic, respectively. In Section 4, we also consider model extensions by assuming that either the interest rate or the default intensity or both are stochastic. In Section 5, we use simulated data to test the impact of the different model assumptions on the cumulative hedging costs. Section 6 concludes the paper. All technical proofs are given in Appendix A.

2 Hedging by Sequential Regression

In incomplete financial markets not every contingent claim is replicable. For this reason a lot of different hedging strategies have been evolved in the literature. On the one hand there exist hedging approaches searching self-financing strategies which reproduce the derivative at the best. On the other hand there are hedging strategies replicating the derivative exactly at maturity by taking into account additional costs during the trading period. While the first class of hedging strategies optimizes the *hedging error*, to be more precisely the difference between the pay-off of the derivative F_T and the liquidation value of the hedging strategy, the other class minimizes the *hedging costs*. In a discrete time set-up Föllmer and Schweizer (1989) developed a hedging approach of the latter type, the so-called *locally risk-minimizing hedging*.

Table 1: *Hedging Concepts: An Overview.*

	Complete Financial Market	Incomplete Financial Market	
No Shortfall	*Delta-Hedging* Black, Merton, Scholes (1973)	*Superhedging* Naik and Uppal (1992)	No Restriction on Initial Costs
Shortfall Risk		*Risk- & Variance-Minimizing Hedging* Föllmer and Sondermann (1986) *Locally Risk-Minimizing Hedging* Föllmer and Schweizer (1989)	
	Globally Risk- and Variance-Minimizing Hedging Schweizer (1995)		
	Shortfall-Hedging Föllmer and Leukert (1999) *(Global) Expected Shortfall-Hedging* Föllmer and Leukert (2000) *Local Expected Shortfall-Hedging* Schulmerich (2001), Schulmerich and Trautmann (2003)		Restriction on Initial Costs

When using two hedging instruments, the underlying asset with price process S and the money market account with price process B, $\mathbf{H} = (h^S, h^B)$ describes the hedging strategy composed of h^S shares in the underlying and h^B shares in the money market account. In a discrete-time setting $\mathrm{V}_t(\mathbf{H}) = h^S_{t+1} S_t + h^B_{t+1} B_t$ denotes the *liquidation value* of the strategy, $G_t(\mathbf{H}) = \sum_{i=1}^{t} (h^S_i \Delta S_i + h^B_i \Delta B_i)$ the *cumulated gain* and finally $\mathcal{C}_t(\mathbf{H}) = \mathrm{V}_t(\mathbf{H}) - G_t(\mathbf{H})$ the cumulated *hedging costs* at time t. To achieve a locally risk-minimizing hedging strategy, Föllmer and Schweizer (1989) solve the following

Problem 1 (Locally risk-minimizing hedging in discrete time)
Search the trading strategy $\mathbf{H}$ *which replicates exactly the derivative* F *at maturity* T *and in addition minimizes the expected quadratic growth of the hedging cost at every point in time:*

$$E^P\left[(\Delta\mathcal{C}_t(\mathbf{H}))^2\,|\mathcal{G}_{t-1}\right] \to \min \ \textit{ for all } t = 1,\ldots,T \textit{ and } \mathbf{H} \in \mathbb{H} \textit{ with } V_T(\mathbf{H}) = F_T\ .$$

A solution of Problem 1 is called *locally risk-minimizing hedging strategy* or *LRM-hedge*[3]. Föllmer and Schweizer (1989) have pointed out that Problem 1 is a sequential regression task that can be solved by backwards induction: In a first step we determine h_T^S and h_T^B by identifying the solution of the subproblem

$$\mathrm{E}^P\left[(\Delta\mathcal{C}_t(\mathbf{H}))^2|\mathcal{G}_{t-1}\right] \to \min \quad \text{for all} \quad h_t^S,\, h_t^B \quad \text{given} \quad \mathrm{V}_t(\mathbf{H}) \tag{1}$$

for $t = T$ with $\mathrm{V}_T(\mathbf{H}) = F_T$. Since we have $\mathrm{V}_t(\mathbf{H}) = h_{t+1}^S S_t + h_{t+1}^B B_t$ for all dates $t = 0,\ldots,T-1$ we know $\mathrm{V}_{T-1}(\mathbf{H})$ and then we can solve the subproblem (1) for $t = T-1$ and thus obtain h_{T-1}^S (as slope of the regression line) and h_{T-1}^B (as intercept), and so on. Since $\Delta\mathcal{C}_t(\mathbf{H}) = \mathrm{V}_t(\mathbf{H}) - (h_t^S S_t + h_t^B B_t)$ holds, (1) is a linear regression problem which can be solved by the least square approach. Figure 2 illustrates this idea.

In the following, we show that this relation shows directly that two different ways of modeling recovery payments lead to the same locally risk-minimizing strategy when hedging a short position in credit derivatives. The first kind of recovery model assumes that the recovery rate is *single-stochastic* since it only depends on the default-time and perhaps the interest rate level as illustrated in part (a) of Figure 1 for a two period set-up. Thus, the recovery amount depends only on the time of default (and the interest rate level).

In the second kind of recovery model the recovery rate is called *doubly-stochastic* allowing in addition (to the default time and the term structure) other risk factors to influence the recovery payment (see part (b) of Figure 1). For example these additional factors can characterize the uncertain costs of financial distress or the uncertain time delay of the promised recovery payment. Thus in this model the default time and the interest rate level do not uniquely determine the recovery payment.

[3] An LRM-hedge also solves the problem

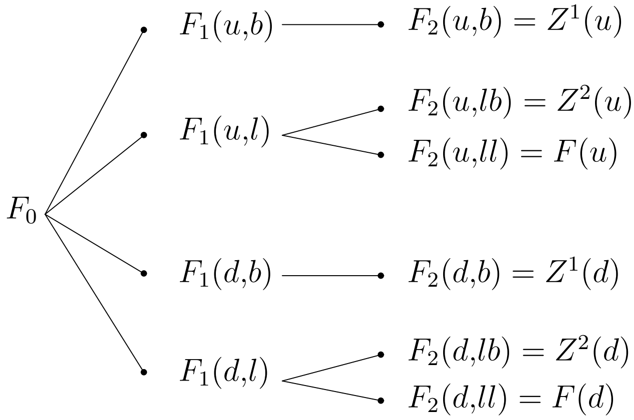
$$\mathrm{E}^P\left[(\Delta C_t(\mathbf{H}))^2\,|\mathcal{G}_{t-1}\right] \to \min \ \text{ for all } t = 1,\ldots,T \text{ and } \mathbf{H} \in \mathbb{H} \text{ with } \mathrm{V}_T(\mathbf{H}) = F_T\ ,$$

where $\Delta C_t(\mathbf{H}) = \Delta\mathcal{C}_t(\mathbf{H})/B_t$ denotes the discounted growth of the hedging costs and B_t is the value of the money market account at time t.

Figure 1: *Single-stochastic versus doubly-stochastic recovery.*

Part (a) of this figure depicts the price process of a credit derivative with a recovery payment depending only on the default time ("l" denotes liquidity, "b" bankruptcy) and the term structure ("u" denotes an up-tick and "d" a down-tick of the interest rate). Conditional on default (and the given term structure) the recovery payment is known. The latter is not the case if the recovery payment is doubly-stochastic. Part (b) of the figure shows that conditional on default (and the given term structure) the recovery payment can take on m different values $Z^1, \ldots, Z^m$.

(a) Price process when recovery is single-stochastic.



(b) Price process when recovery is doubly-stochastic.

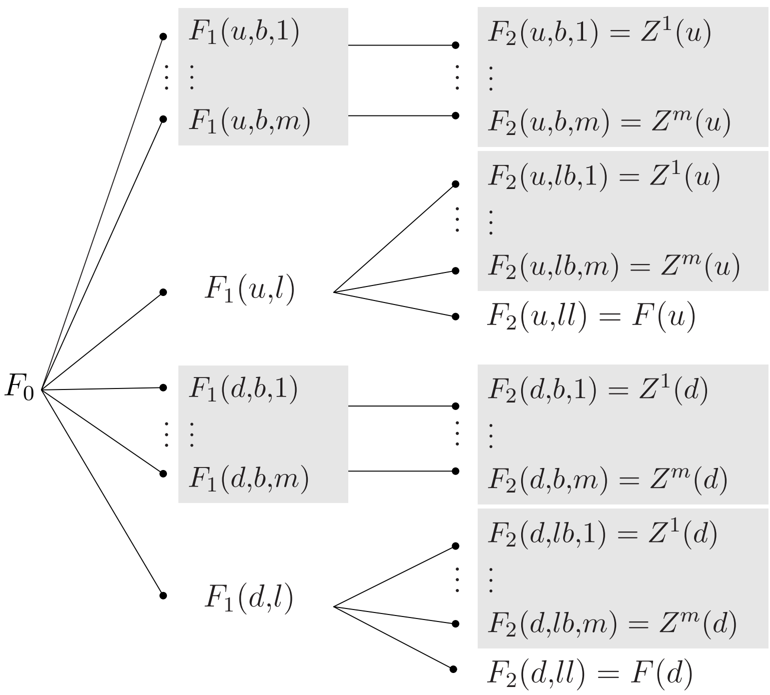
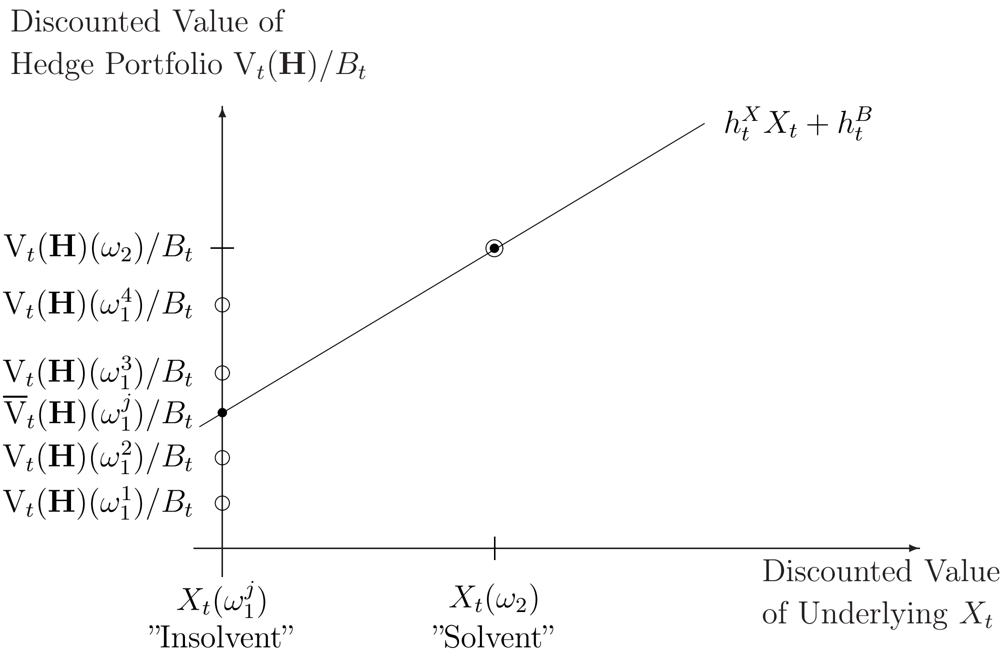


Figure 2 already illustrates the key result of this paper: the locally risk-minimizing hedging strategy for the credit derivative is the same for single- and doubly-stochastic recovery modeling, provided that the *expected* doubly-stochastic recovery payment conditional on the default time (and the term structure) coincides with the single-stochastic recovery payment conditional on the default time (and the interest rate level).

Figure 2: *LRM-strategy when recovery is doubly-stochastic.*

When recovery is doubly-stochastic the payment at default does not only depend on the default time and the interest rate level but also on another risk factor. Different realizations of this risk factor are denoted by the superscript j in the state ω_i^j where the subscript i denotes different states of the world influencing the underlying instrument. Since the underlying (say, shares of common stock of the firm, or a corporate zero-bond with total loss at default written on the underlying firm) does not depend on the additional factor, its discounted price is always zero at default, $X_t(\omega_1^1) = X_t(\omega_1^2) = \ldots = 0$. The symbol " $\circ$ " describes a possible realization of the discounted value of the hedge portfolio. To determine the LRM-hedge we have to run a regression for the five value tuples represented by the $\circ$-symbol. Alternatively, we can calculate in a first step the average value of the hedge portfolio $\overline{\mathrm{V}}_t(\mathbf{H})(\omega_1^1)/B_t = \overline{\mathrm{V}}_t(\mathbf{H})(\omega_1^2)/B_t = \ldots$, conditional on the default event occurring. The latter pairs of values are denoted with the " $\bullet$ " symbol. In a second step, we identify the slope for the regression line for the points " $\bullet$ " (only two tuples, as you can see) which equals the slope of the first regression.



The insight provided by Figure 2 can be proven in a more formal way. We show that the single-stage regression approach (delivers the LRM-hedge of a defaultable claim assuming doubly-stochastic recovery) and a two-stage procedure (delivers the LRM-hedge of a defaultable claim assuming single-stochastic recovery which coincides at any default time with the expectation of the doubly-stochastic recovery conditional on the default time) provide the same result. With the conventions $p_i = \sum_j p(\omega_i^j)$, $\overline{X}_t(\omega_i^j) = \sum_k X_t(\omega_i^k)p(\omega_i^k)/p_i$, and $\overline{\mathrm{V}}_t(\mathbf{H})(\omega_i^j) = \sum_k \mathrm{V}_t(\mathbf{H})(\omega_i^k)p(\omega_i^k)/p_i$ for all j, we obtain

$$\mathrm{E}^P[\mathrm{V}_t(\mathbf{H})|\mathcal{G}_{t-1}] = \sum_{i,k} p(\omega_i^k)\mathrm{V}_t(\mathbf{H})(\omega_i^k) = \sum_i p_i\overline{\mathrm{V}}_t(\mathbf{H})(\omega_i^j) = \mathrm{E}^P[\overline{\mathrm{V}}_t(\mathbf{H})|\mathcal{G}_{t-1}] \,,$$

and in an analogous manner $\mathrm{E}^P[(X_t)^2|\mathcal{G}_{t-1}] = \mathrm{E}^P\left[(\overline{X}_t)^2|\mathcal{G}_{t-1}\right]$, $\mathrm{E}^P[X_t|\mathcal{G}_{t-1}] = \mathrm{E}^P[\overline{X}_t|\mathcal{G}_{t-1}]$, $\mathrm{E}^P[\mathrm{V}_t(\mathbf{H})X_t|\mathcal{G}_{t-1}] = \mathrm{E}^P[\overline{X}_t\overline{\mathrm{V}}_t(\mathbf{H})|\mathcal{G}_{t-1}]$. From this, it follows that the hedge ratio (slope of the regression line) and the shares in the money market account (intercept of the regression line) of the one-stage regression approach,

$$h_t^S = \frac{\mathrm{Cov}^P[\mathrm{V}_t(\mathbf{H}),X_t|\mathcal{G}_{t-1}]}{\mathrm{Var}^P[X_t|\mathcal{G}_{t-1}]B_t} \quad \text{and} \quad h_t^B = \frac{\mathrm{E}^P[\mathrm{V}_t(\mathbf{H})|\mathcal{G}_{t-1}]}{B_t} - h_t^S\mathrm{E}^P[X_t|\mathcal{G}_{t-1}] \,,$$

coincide with these of the two-stage procedure:

$$\overline{h_t^S} = \frac{\mathrm{Cov}^P[\overline{\mathrm{V}}_t(\mathbf{H}),\overline{X}_t|\mathcal{G}_{t-1}]}{\mathrm{Var}^P[\overline{X}_t|\mathcal{G}_{t-1}]B_t} \quad \text{and} \quad \overline{h_t^B} = \frac{\mathrm{E}^P[\overline{\mathrm{V}}_t(\mathbf{H})|\mathcal{G}_{t-1}]}{B_t} - \overline{h_t^S}\mathrm{E}^P[\overline{X}_t|\mathcal{G}_{t-1}] \,.$$

3 Hedging in Reduced-Form Models

Below we will determine hedging strategies for credit derivatives, e.g. defaultable bonds and credit default swaps. We envision a situation where a hedger owns a portfolio of such credit derivatives and tries to hedge this portfolio against all kinds of risk, namely default risk, interest rate risk and recovery rate risk. Suitable hedging instruments are then money market accounts, CDSs, junior bonds and so on.

In the following we assume that the hedger tries to hedge a short position in a coupon-paying defaultable bond. This defaultable bond delivers time-continuous cash flows C_t in $0 \leq t \leq T$ as long as no default has occurred. If the firm is still solvent at the time of maturity a payment F will also be paid. Otherwise the owner of the credit derivative receives (in addition to the cash flow stream C during the period $[0,\tau)$) the *uncertain* recovery payment $Z(\tau)$ depending on default time $t = \tau$ and paid out at $t = T$. We denote the defaultable coupon bond by (Z, C, F).[4] We assume that the recovery amount does not exceed the remaining value of the credit derivative's cash flow when no default occurs:

$$0 \leq Z(\tau) \leq B_T \int_\tau^T C_t/B_t \, dt + F \quad P\text{-a.s.}, \tag{2}$$

where $B_t = \exp\{\int_0^t r_s \, ds\}$ denotes the value of the money market account at time t and P denotes the statistical probability measure. At any time $t < \tau$, the recovery payment for a credit event occurring at time $\tau = u$ has an expected value of $\mu^Z(u)$ and a standard deviation of $\sigma^Z(u)$ under P. Because of (2) we have also

$$0 \leq \mu^Z(u) \leq B_T \int_u^T C_t/B_t \, dt + F \quad \text{for} \quad 0 < u \leq T.$$

For technical reasons we assume $\sup_{u \in [0,T]} \sigma^Z(u) < \infty$. Assumption (2) guarantees that the value of the defaultable claim (Z, C, F) is lower than the value of a default-

[4]When hedging a CDS, we have a different hedging situation. In this case, one hedges a claim of the form $(F - Z, \, -C, 0)$, since a CDS pays the difference between the recovery payment and the promised face value, $F - Z$, and the buyer of the CDS does not receive but has to pay a time-continuous premium.

free but otherwise identical derivative (C,F).
The cumulative value of the credit derivative at maturity amounts to

$$F_T = \begin{cases} B_T \int_0^T C_t/B_t \, dt + F \ , & \text{if } \tau > T \\ B_T \int_0^\tau C_t/B_t \, dt + Z(\tau) \ , & \text{if } \tau \le T \end{cases} .$$

The stochastic *recovery rate* of the credit derivative (Z,C,F) is defined as follows:

$$\delta(\tau) = \frac{B_T \int_0^\tau C_t/B_t \, dt + Z(\tau)}{B_T \int_0^T C_t/B_t \, dt + F} = \frac{\widetilde{C}_\tau \cdot B_T + Z(\tau)}{\widetilde{C}_T \cdot B_T + F} \quad \in \quad [0,1], \tag{3}$$

where $\widetilde{C}_t = \int_0^t C_s/B_s \, ds$ denotes the present value of the cash flow stream C during $[0,t]$ when default has not occurred until t. Relation (3) relates the final value of the defaultable claim's cash flows (Z,C,F) to the final value of the default-free, but otherwise identical derivative's cash flows (C,F).[5] Because of assumption (2) the recovery rate is lower than one. If the recovery only depends on the uncertain default time and the interest level, we will call it *single-stochastic*. If it is subject to another source of risk, we will denote the recovery *doubly-stochastic*.

We assume that the seller of this defaultable claim (Z,C,F) can hedge his short position with strategy $\mathbf{H} = (h^X, h^B)$ consisting of h^X defaultable zero bonds with total loss in case of default and h^B money market accounts. To simplify the following presentation we start with a *deterministic* term structure, i.e. the short rate $(r_t)_{t\in[0,T]}$ is a deterministic function of time.

3.1 A Simple Intensity Model

This section presents a simple intensity model in continuous time which describes a possible default of a firm at time $\tau > 0$ during the time horizon $[0,T]$. Trading takes place every time $t \in [0,T]$. The credit event is specified in terms of an exogenous jump process, the so-called *default process* $H_t = \mathbb{1}_{\{\tau \le t\}}$. In the following we assume that H is an inhomogeneous Poisson process stopped at the first jump – the default time:

$$P(\tau \le t) = P(H_t = 1) = 1 - \exp\left\{ -\int_0^t \lambda(s) \, ds \right\} \quad \text{for every} \quad t \ge 0 \ ,$$

[5]Bakshi, Madan and Zhang (2006, p. 22) define the recovery rate by means of the out-standing payments. But the definition above simplifies the following formulae for the hedging strategies.

where λ is a deterministic, non-negative function of time with $\int_0^T \lambda(t)\,dt < \infty$ representing the *default intensity* under the statistical probability measure P. The model is based on a probability space $(\Omega,\mathcal{G},P)$, where Ω denotes the state space in the economy. The information available to the market participants at time t is given by the filtration $(\mathcal{G}_t)_{t\in[0,T]}$ generated by the marked inhomogeneous Poisson process $H^Z = (H,Z)$ stopped at the first jump: $\mathcal{G}_t = \sigma(H_t^Z)$ for $t \in [0,T]$. $X = (X_t)_{t\in[0,T]}$ denotes the discounted price process of the traded defaultable zero coupon bond with maturity date T *and total loss in case of default* given by

$$X_t = \frac{1}{B_T}\exp\left\{-\int_t^T \widehat{\lambda}(s)\,ds\right\}(1-H_t) \tag{4}$$

if financial markets are frictionless and arbitrage-free. The deterministic non-negative function $\widehat{\lambda}$ with $\int_0^T \widehat{\lambda}(t)\,dt < \infty$ can be estimated via market values of defaultable financial instruments[6] and specifies the default intensity under the martingale measure $Q \in \mathbb{Q}$. In particular,

$$\begin{aligned} \mathrm{E}^Q[X_t|\mathcal{G}_s] &= \mathbb{1}_{\{\tau>s\}}\left(X_t \cdot Q(\tau>t|\tau>s) + 0\cdot Q(\tau\le t|\tau>s)\right) \\ &= (1-H_s)\frac{1}{B_T}\exp\left\{-\int_t^T \widehat{\lambda}(u)\,du\right\}\exp\left\{-\int_s^t \widehat{\lambda}(u)\,du\right\} = X_s\,. \end{aligned}$$

The discounted price process X admits the decomposition $X = X_0 + A + M$, since

$$\begin{aligned} dX_t = \widehat{\lambda}(t)X_{t-}dt - X_{t-}dH_t &= X_{t-}(\widehat{\lambda}(t)-\lambda(t))dt - X_{t-}d\widetilde{H}_t \\ &= dA_t + dM_t\,. \end{aligned}$$

Here, $\widetilde{H}_t = H_t - \int_0^{t\wedge\tau}\lambda(s)\,ds$ denotes the *compensated default process*, A describes the continuous drift component with $A_0 = 0$, M denotes a square integrable P-martingale[7] with $M_0 = 0$, and finally $X_0 = \exp\left\{-\int_0^T \widehat{\lambda}(s)\,ds\right\}/B_T$ denotes the bond price at $t = 0$. Due to properties of the conditional quadratic variation (see, e.g., Protter (1990)) it follows that

$$d\langle M\rangle_t = X_{t-}^2\, d\langle \widetilde{H}\rangle_t = X_{t-}^2\lambda(t)d(t\wedge\tau) = X_{t\wedge\tau-}^2\lambda(t)d(t\wedge\tau)\,.$$

Since $dA_t = X_{t-}(\widehat{\lambda}(t)-\lambda(t))dt = X_{t\wedge\tau-}(\widehat{\lambda}(t)-\lambda(t))d(t\wedge\tau)$ we obtain

$$A_t = \int_0^t \widetilde{\alpha}_s\,d\langle M\rangle_s \quad\text{with}\quad \widetilde{\alpha}_t = \frac{1}{X_{t\wedge\tau-}}\left(\frac{\widehat{\lambda}(t)}{\lambda(t)} - 1\right),$$

[6] See, e.g., Jarrow and Turnbull (1995) and Jarrow, Lando and Turnbull (1997).

[7] Since the process $\widetilde{H}$ is a square integrable martingale with $[\widetilde{H},\widetilde{H}] = H$ and since the process X_- is predictable with $\mathrm{E}^P[\int_0^T X_{t-}^2\, d[\widetilde{H},\widetilde{H}]_t] = \mathrm{E}^P[\int_0^T X_{t-}^2\, dH_t] < \infty$, M is also a square integrable martingale (see Protter, 1990, p. 142).

and therefore $X = X_0 + \int \widetilde{\alpha}\, d\langle M\rangle + M$.

Now we determine hedging strategies for defaultable claims which minimize the risk locally. More precisely, we solve Problem 2 as stated in the appendix. This rather technical formulation is due to Schweizer (1991) and can be seen as continuous-time analogue of Problem 1. To identify the LRM-hedge for credit derivatives we use the minimal martingale measure[8] $\widehat{P}$ defined by the density [9]

$$\begin{aligned}
\widehat{Z}_t &= \mathcal{E}\left\{-\int \widetilde{\alpha}\, dM\right\}_t \\
&= \mathcal{E}\left\{\int_0^{t\wedge\tau} \left(\lambda(s) - \widehat{\lambda}(s)\right) ds + \left(\frac{\widehat{\lambda}(\tau)}{\lambda(\tau)} - 1\right) H_t\right\} \\
&= \begin{cases} \exp\{\int_0^t (\lambda(s) - \widehat{\lambda}(s))\, ds\}\,, & \text{if } t < \tau, \\ \frac{\widehat{\lambda}(\tau)}{\lambda(\tau)} \exp\{\int_0^\tau (\lambda(s) - \widehat{\lambda}(s))\, ds\}\,, & \text{if } t \geq \tau. \end{cases} \qquad (5)
\end{aligned}$$

Thus the distribution of the recovery payment remains unaffected by the measure change and the default intensity under $\widehat{P}$ is given by $\widehat{\lambda}$.[10]

The discounted value of the recovery payment under the assumption that the stock price jumps to/or reaches a pre-specified value when the credit event occurs time t, conditional on the event that default takes place in $(t,T]$ is given by the deterministic function

$$\begin{aligned}
g_t^Z &= \widehat{E}\left[\frac{1}{B_T}\mu^Z(\tau)\mathbb{1}_{\{\tau\leq T\}} \middle| \tau > t\right] \\
&= \frac{1}{B_T}\widehat{E}\left[\mu^Z(\tau)\mathbb{1}_{\{\tau\leq T\}} \middle| t < \tau \leq T\right] \cdot \widehat{P}(\tau \leq T | \tau > t) \\
&= \frac{1}{B_T}\int_t^T \exp\left\{-\int_t^u \widehat{\lambda}(s)\, ds\right\} \widehat{\lambda}(u)\mu^Z(u)\, du. \qquad (6)
\end{aligned}$$

Likewise, the discounted value g^F of the payment F being paid out in case of no default up to time T and the discounted value g^C of the future coupon payments of

[8]The notion "minimal martingale measure" is motivated by the fact that apart from turning X into a martingale this measure disturbs the overall martingale and orthogonality structures as little as possible.

[9]For evaluating the stochastic exponential see, e.g., Protter (1990, p. 77).

[10]More precisely, from Theorem $T2$ in Brémaud (1981, p. 165f.) it follows that (5) coincides with the density corresponding to the measure change from P to Q. Hence, we have $\widehat{P} \equiv Q$.

the credit derivative being paid out until the time of default are, respectively, given by

$$g_t^F = \frac{1}{B_T} \exp\left\{-\int_t^T \widehat{\lambda}(s)\, ds\right\} \cdot F, \tag{7}$$

$$g_t^C = \int_t^T \frac{C_u}{B_u} \exp\left\{-\int_t^u \widehat{\lambda}(s)\, ds\right\} du. \tag{8}$$

Due to the results of Schweizer (1991) and with the convention

$$V_t^F = \widehat{E}[F_T/B_T|\mathcal{G}_t]$$

Lemma 1 provides the LRM-hedge ratio via the Föllmer-Schweizer-decomposition, see Föllmer and Schweizer (1991).

Lemma 1 (FS-Decomposition of a Credit Derivative)
The discounted cumulative value F_T/B_T of the credit derivative (Z, C, F) at maturity has the following strong Föllmer-Schweizer-decomposition:

$$V_T^F = F_T/B_T = F_0 + \int_0^T h_t^X\, dX_t + L_T^F\ ,$$

where

$$h_t^X = \frac{d\langle V^F, X\rangle_t}{d\langle X\ , X\rangle_t} = \begin{cases} \frac{g_{t-}^C + g_{t-}^F + g_{t-}^Z}{X_{t-}} - \frac{\mu^Z(t)}{B_T X_{t-}} & : \ t \leq \tau, \\ 0 & : \ t > \tau, \end{cases}$$

is the locally risk-minimizing hedge ratio, $F_0 = g_0^C + g_0^F + g_0^Z$ is a constant, and L^F is a martingale which is orthogonal to M, given by $L_t^F = \int_0^t \frac{1}{B_T}(Z(s) - \mu^Z(s))\, d\widetilde{H}_s$.

3.2 Single-Stochastic Recovery Payment

We first consider the case of a single-stochastic recovery payment, i.e. $Z(t)$ is a deterministic function of time. The discounted recovery payment expected at time t under the martingale measure Q, given the credit event takes place in $(t,T]$ is represented by the deterministic function

$$g_t^Z = \mathrm{E}^Q\left[\frac{1}{B_T}\mu^Z(\tau)\mathbb{1}_{\{\tau \leq T\}}|\tau > t\right] = \frac{1}{B_T}\int_t^T \exp\left\{-\int_t^u \widehat{\lambda}(s)\, ds\right\} \widehat{\lambda}(u) Z(u)\, du. \tag{9}$$

Replacing $\mu^Z(t)$ by $Z(t)$ in Lemma 1 results in

Proposition 1 (Replication for Single-Stochastic Recovery)
The credit derivative (Z,C,F) *with single-stochastic recovery is duplicated by the hedging strategy* $\mathbf{H} = (h^X,h^B)$ *with*

$$\begin{aligned} h_t^X &= \frac{g_{t-}^C + g_{t-}^F + g_{t-}^Z}{X_{t-}} - \frac{Z(t)}{B_T X_{t-}}\,, \\ h_t^B &= V_t(\mathbf{H})/B_t - h_t^X X_{t-} = \widetilde{C}_t + Z(t)/B_T, \end{aligned}$$

for $t \leq \tau$*, and* $h_t^X = 0$, $h_t^B = h_\tau^B$ *for* $t > \tau$*.*

According to this duplication strategy at every point in time t the value of the money market accounts equals the cumulative value of the credit derivative in the case of default at time $\tau = t$. The value of the position in the defaultable zeros at time $t < \tau$ equals the discounted expected future payments of the credit derivative less the discounted recovery payment in the case of default at time t, i.e.

$$h_t^X X_t = g_{t-}^C + g_{t-}^F + g_{t-}^Z - \frac{Z(t)}{B_T}.$$

It is worth mentioning that, see Müller (2008), in the special case when the recovery rate is *constant*, $\delta(u) = \delta$ for all default times $\tau = u$, the expected recovery rate given that default occurs in $(t,T]$, denoted by $\widetilde{\mu}^\delta(t)$, is given by

$$\begin{aligned} \widetilde{\mu}^\delta(t) &= \delta \int_t^T \widehat{\lambda}(u) \exp\left\{ - \int_t^u \widehat{\lambda}(s)ds \right\} du \\ &= \delta \left[- \exp\left\{ - \int_t^u \widehat{\lambda}(s)ds \right\} \right]_t^T \\ &= \delta(1 - X_t B_T), \end{aligned} \tag{10}$$

and it will then be possible to replicate the credit derivative (Z,C,F) with single-stochastic recovery by a *static* hedge: Buy

$$h^X = (1-\delta)(\widetilde{C}_T B_T + F)$$

defaultable zero bonds (with total loss in case of default) and buy

$$h^B = \delta(\widetilde{C}_T + F/B_T)$$

money market accounts.

3.3 Doubly-Stochastic Recovery Payment

We now consider the case of a doubly-stochastic recovery payment, i.e. Z now is a stochastic process. Every probability measure $Q \in \mathbb{Q}$ with corresponding default intensity $\widehat{\lambda}$ and arbitrary distribution of the recovery rate with values in [0,1] represents an equivalent martingale measure if the null sets of the distribution of the recovery rate under Q and P are the same. The financial market will be arbitrage-free. But it will be incomplete if the recovery rate is not known P-a.s. given that default occurs in $\tau = t$. For this reason defaultable claims with a doubly-stochastic recovery can *not* be duplicated. The incompleteness of the financial market model can also be recognized as follows: There are two sources of risk – the default time and the amount of the recovery are uncertain. But there exists only one financial instrument (besides the money market account) for hedging the default risk.

Proposition 2 (LRM-Hedge)
The locally risk-minimizing hedge of the credit derivative (Z, C, F) amounts to

$$\begin{aligned} h_t^X &= \frac{g_{t-}^C + g_{t-}^F + g_{t-}^Z}{X_{t-}} - \frac{\mu^Z(t)}{B_T X_{t-}} , \\ h_t^B &= V_{t-}^F - h_t^X X_{t-} = \widetilde{C}_t + \mu^Z(t)/B_T; . \end{aligned}$$

After default, i.e. for $t > \tau$, we have

$$h_t^X = 0 , \quad h_t^B = \widetilde{C}_\tau + Z(\tau)/B_T .$$

In the case of a defaultable claim with single-stochastic recovery the locally risk-minimizing hedge collapses to the duplication strategy given in Proposition 1.

According to this duplication strategy at every point in time t the value of the money market accounts equals the cumulative value of the credit derivative in the case of default at time $\tau = t$. At default the share in the money market account makes a jump in the amount of $(Z(\tau) - \mu^Z(\tau))/B_T$ such that the value of the hedging strategy at maturity coincides with the discounted cumulative value of the credit derivative. The value of the position in the defaultable zeros at time $t < \tau$ equals the discounted expected future payments of the credit derivative less the discounted expected recovery payment in the case of default at time t, i.e.

$$h_t^X X_t = g_{t-}^C + g_{t-}^F + g_{t-}^Z - \frac{\mu^Z(t)}{B_T}.$$

Because of the relation $C(\mathbf{H}) = V_0^F + L^F$ the LRM-hedge is self-financing at every point in time before and after default. But at default money accrues or flows out, depending on the difference between realized recovery payment, $Z(\tau)$, and the expected payment at default, $\mu^Z(\tau)$. On average, the locally risk-minimizing hedging strategy is self-financing, that is, the strategy is *mean*-self-financing.

If the recovery is single-stochastic the LRM-hedge will even be self-financing and therefore will collapse to a replication strategy. For the special case, see Müller (2008), that the expected recovery rate does not depend on the default time, i.e. $\mu^\delta(u) = \mu^\delta$ at $0 < u \leq T$, and hence $\widetilde{\mu}^\delta(t) = \mu^\delta(1 - B_T X_{t-})$ for $t \leq \tau$, the locally risk-minimizing hedge simplifies to a static hedge:

$$\mathbf{H} = (h^X, h^B) = (\,(\widetilde{C}_T B_T + F)(1 - \mu^\delta), (\widetilde{C}_T + F/B_T)\mu^\delta\,)\;.$$

Proposition 2 shows that the locally risk-minimizing hedge depends only on the expected payment at default under the statistical probability measure, but not on other details of the probability distribution of the recovery. Hence we achieve the following result:

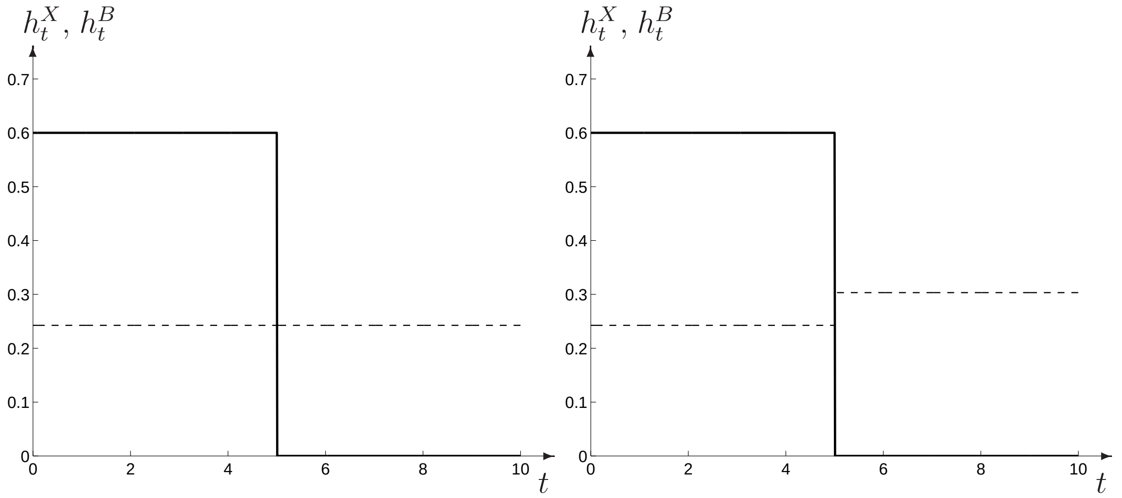
Proposition 3 (Impact of Recovery Modeling on LRM-Hedge)
The LRM-hedge for a credit derivative (Z^d, C, F) with a doubly-stochastic recovery equals the LRM-hedge for a defaultable claim (Z^s, C, F) with single-stochastic recovery for all points in time until default, provided that the expected recovery payments coincide under the statistical probability measure, i.e. $\mu^{Z^d}(u) = \mu^{Z^s}(u) = Z^s(u)$ for every $0 < u \leq T$.

Example 1 We consider a financial market where a defaultable zero bond of a firm with total loss at default and maturity 10 years is traded. Furthermore, we assume a flat term structure with $r = 5$ %. Default time is exponentially distributed with intensity $\lambda = 0{,}05$ and $\widehat{\lambda} = 0{,}20$ under the statistical probability measure and the martingale measure, respectively. We now calculate hedging strategies of a defaultable zero bond with recovery payment at default. We assume a single-stochastic, even *constant* recovery rate of $\delta^s = 40$ %, and a doubly-stochastic recovery rate with an expected value of $\mu^{\delta^d} = 40$ %.

Figure 3 shows the locally risk-minimizing hedging strategy of a zero with single- and doubly-stochastic recovery. We assume, that the firm defaults after 5 years and

Figure 3: *LRM-hedges when default occurs at $\tau = 5$*

The left figure illustrates the LRM-hedge for a defaultable zero bond with *constant* recovery. This hedge corresponds to the duplication. The right figure depicts the LRM-hedge for a defaultable zero bond with an *uncertain* recovery payment when default occurs after five years with a realized recovery rate of 50 %. The solid line describes the hedge ratio h^X and the dashed line the number of money market accounts h^B during time.



that the realized recovery rate amounts to 50 % in the case of doubly-stochastic recovery modeling. According to Proposition 3 the LRM-hedges are equal until default for both the single- and the doubly-stochastic recovery case. After the credit event the shares in the money market account of the locally risk-minimizing strategies differ since the realised payments at default are different.

If an investor prefers a self-financing hedging strategy, the so-called *super-hedging strategy* which assures a liquidation value at maturity at least as high as the pay-off of the derivative, i.e. $\mathrm{V}_T(\mathbf{H}) \geq F_T$ P-a.s., then the recovery modeling has the following impact on the hedging strategy. Assuming a constant recovery payment of 0,40 the super-hedge corresponds to the duplication strategy $\mathbf{H} = (h^X, h^B) = (0{,}60;\ 0{,}40/B_T)$ as well as the LRM-hedge. If the payment at default is uncertain, the super-hedge depends on the distribution of the recovery, more precisely, on the domain of the recovery payment. Assuming that the recovery payment can reach values on $[0, 1]$ and $[0, 0{,}95]$, respectively, the resulting super-hedges are $\mathbf{H} = (h^X, h^B) = (0;\ 1/B_T)$ and $\mathbf{H} = (h^X, h^B) = (0{,}05;\ 0{,}95/B_T)$, respectively. $\square$

4 Extensions

Closed-form solutions of hedging strategies for credit derivatives are rare in the literature. For instance, Bielecki et al. (2008) prove the existence of a hedging strategy for a credit derivative (Z, C, F) in a general setup (including both stochastic interest and stochastic default rates), but do not provide the hedge ratio in closed-form. Biagini and Cretarola (2009) derive locally risk-minimizing strategies, but give closed-form solutions only for the special case of null interest rates, no coupon payments and a predictable, hence single-stochastic recovery payment.

So far, the recovery payment and the time of default were the only random quantities in our model as well. In Section 4.1, we derive the LRM-strategy in case the interest rate is also stochastic while in Section 4.2 we consider the case of a stochastic default intensity instead. In Section 4.3, r and $\widehat{\lambda}$ are then assumed to be both stochastic but independent. This independence assumption, however, will turn out to be no major restriction.

4.1 Stochastic Interest Rates

We now extend our basic model to the case of a non-trivial reference filtration to investigate to what extent the hedging strategy will be affected. Due to this additional source of risk, we now have $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$, where $\mathcal{F}_t$ describes the time-t information about the evolution of the interest rate and the default rate and $\mathcal{H}_t$ describes the time-t market information about whether default has occured and the recovery risk. In particular, we assume $\mathcal{F}_t = \sigma(W_t)$ for some Brownian motion W.

Consider the $(\mathcal{F}_t)$-martingale

$$\begin{aligned} m_t &= \widehat{E}\left[\frac{1}{B_T}\int_0^T \exp\left\{-\int_t^u \widehat{\lambda}(s)\,ds\right\}\widehat{\lambda}(u)\mu^Z(u)\,du \right. \\ &\qquad \left. + \exp\left\{-\int_0^T \widehat{\lambda}(s)\,ds\right\}\cdot\frac{F}{B_T} + \int_0^T \frac{1}{B_u}\exp\left\{-\int_t^u \widehat{\lambda}(s)\,ds\right\}\,dC_u \middle| \mathcal{F}_t\right]. \end{aligned}$$

Denote by ξ the predictable process appearing in the martingale representation of the process m, i.e.

$$m_t = m_0 + \int_0^t \xi_s\,d\widehat{W}_s, \tag{11}$$

Lemma 2 provides the LRM-hedge ratio via the Föllmer-Schweizer-decomposition in case the reference filtration $(\mathcal{F}_t)$ is non-trivial.

Lemma 2 (FS-Decomposition in case of a Brownian Reference Filtration) *The discounted cumulative value F_T/B_T of the credit derivative (Z, C, F) at maturity has the following strong Föllmer-Schweizer-decomposition:*

$$V_T^F = F_T/B_T = F_0 + \int_0^T h_t^X \, dX_t + L_T^F \,,$$

where

$$\begin{aligned} h_t^X &= \frac{d\langle V^F, X\rangle_t}{d\langle X\ , X\rangle_t} \\ &= (1 - H_t) \left(\exp\left\{ \int_0^t \widehat{\lambda}(s) \, ds \right\} \cdot \frac{\xi_t}{\sigma(t) X_{t-}} + \frac{g_{t-}^C + g_{t-}^F + g_{t-}^Z}{X_{t-}} - \frac{\mu^Z(t)}{\widehat{E}[B_T|\mathcal{F}_t] X_{t-}} \right) , \end{aligned}$$

is the locally risk-minimizing hedge ratio, $F_0 = g_0^C + g_0^F + g_0^Z$ is a constant, and L^F is a martingale which is orthogonal to M, given by $L_t^F = \int_0^t \frac{1}{B_T} (Z(s) - \mu^Z(s)) \, d\widetilde{H}_s$.

Proposition 4 (LRM-Hedge in case of a Brownian Reference Filtration) *In case of stochastic interest rates, the locally risk-minimizing hedging strategy of the credit derivative (Z,C,F) is given by*

$$\begin{aligned} h_t^X &= \exp\left\{ \int_0^t \widehat{\lambda}(s) \, ds \right\} \cdot \frac{\xi_t}{\sigma(t) X_{t-}} + \frac{g_{t-}^C + g_{t-}^F + g_{t-}^Z}{X_{t-}} - \frac{\mu^Z(t)}{\widehat{E}[B_T|\mathcal{F}_t] X_{t-}}, \\ h_t^B &= V_{t-}^F - h_t^S X_{t-}, \end{aligned}$$

for $t \leq \tau$, and

$$\begin{aligned} h_t^X &= 0, \\ h_t^B &= \int_0^\tau \frac{1}{B_s} \, dC_s + \widehat{E} \left[\frac{1}{B_T} | \mathcal{F}_t \right] Z_\tau, \end{aligned}$$

for $t > \tau$.

From Proposition 4 we see that the LRM-hegde will be given explicitly, if we can find an explicit representation of the process ξ.

Suppose now that the interest rate follows a stochastic process while the default rate is a deterministic function. The $\widehat{P}$-dynamics of the defaultable zero bond are then given by

$$dX_t = \widehat{\lambda}(t) X_{t-} dt + \sigma(t) X_{t-} d\widehat{W}_t - X_{t-} dH_t,$$

and we thus have

$$\begin{aligned} d\langle X,X\rangle_t = d\langle M,M\rangle_t &= \sigma^2(t)X_{t-}^2 d\langle W\rangle_t + X_{t-}^2 d\langle \widetilde{H}\rangle_t \\ &= \left(\sigma^2(t) + \widehat{\lambda}(t)\right) X_{t-}^2 dt. \end{aligned}$$

For g^Z, g^F and g^C, we have

$$g_t^Z = \widehat{E}\left[\frac{1}{B_T}|F_t\right]\int_t^T \exp\left\{-\int_t^u \widehat{\lambda}(s)\,ds\right\}\widehat{\lambda}(u)\mu^Z(u)\,du, \tag{12}$$

$$g_t^F = \widehat{E}\left[\frac{1}{B_T}|F_t\right] \exp\left\{-\int_t^T \widehat{\lambda}(s)\,ds\right\}\cdot F, \tag{13}$$

$$g_t^C = \widehat{E}\left[\int_t^T \frac{1}{B_u}\exp\left\{-\int_t^u \widehat{\lambda}(s)\,ds\right\}\,dC_u|\mathcal{F}_t\right], \tag{14}$$

respectively.

Example 2 Suppose the short rate follows the CIR model under the minimal martingale measure, i.e.

$$dr_t = \kappa^r(\theta^r - r_t)dt + \sigma^r\sqrt{r_t}d\widehat{W}_t,$$

where κ^r, θ^r, σ^r, $r_0>0$.

Thus

$$\widehat{E}\left[\frac{1}{B_T}|\mathcal{F}_t\right] = \exp\left\{-\int_0^t r_s\,ds - r_tC(t,T) - D(t,T)\right\}, \tag{15}$$

where,

$$C(t,T) = \frac{\sinh(\gamma^r(T-t))}{\gamma^r\cosh(\gamma^r(T-t)) + \frac{1}{2}\kappa^r\sinh(\gamma^r(T-t))}, \tag{16}$$

$$D(t,T) = -\frac{2\kappa^r}{(\sigma^r)^2}\ln\left(\frac{\gamma^r e^{\frac{1}{2}\kappa^r(T-t)}}{\gamma^r\cosh(\gamma^r(T-t)) + \frac{1}{2}\kappa^r\sinh(\gamma^r(T-t))}\right), \tag{17}$$

$\gamma^r = \frac{1}{2}\sqrt{(\kappa^r)^2 + 2(\sigma^r)^2}$, $\sinh u = \frac{e^u - e^{-u}}{2}$, and $\cosh u = \frac{e^u + e^{-u}}{2}$.

Defining $G_t = \exp\{-\int_0^t \widehat{\lambda}(s)\,ds\}$ for all t, we have

$$\begin{aligned} m_t &= \widehat{E}\left[\frac{1}{B_T}|\mathcal{F}_t\right]\cdot\left(G_T\cdot F + \int_0^t G_s\,\widehat{\lambda}(s)\,\mu^Z(s)\,ds\right) \\ &\quad + \int_0^t \frac{G_s}{B_s}\,dC_s + \int_t^T \widehat{E}\left[\frac{1}{B_s}|\mathcal{F}_t\right]G_s\,dC_s \\ &=: u(t,r_t). \end{aligned}$$

From Proposition A.1, it follows that the process ξ is given by

$$
\begin{aligned}
\xi_t &= \sigma^r \sqrt{r_t}\, \frac{\partial}{\partial r} u(t,r_t) \\
&= \sigma^r \sqrt{r_t} \cdot \Bigg[-C(t,T)\widehat{E}\left[\frac{1}{B_T} \Big| \mathcal{F}_t\right] \left(G_T \cdot F + \int_0^T G_s \, \widehat{\lambda}(s) \, \mu^Z(s) \, ds \right) \\
&\quad - \int_t^T C(t,s)\widehat{E}\left[\frac{1}{B_s} \Big| \mathcal{F}_t\right] G_s \, dC_s \Bigg].
\end{aligned}
$$

Hence, the hedging strategy is given explicitly.

□

Figure 4: *Initial hedge ratio as a function of the interest rate level.*

The figure shows the initial hedge ratio as a function of the interest rate level for an expected recovery payment of $\mu^Z = 20$ (solid lines), $\mu^Z = 50$ (dashed lines) and $\mu^Z = 80$ (dashed-dotted lines). The blue graphs illustrate the case of deterministic interest rates for parameters $t = 0$, $T = 1$, $\widehat{\lambda} = 2$, $C = 7$ and $F = 100$. The red graphs illustrate the case of stochastic interest rates with CIR dynamics for parameters $\kappa^r = 0.5$, $\theta^r = 0.05$ and $\sigma^r = 0.2$.

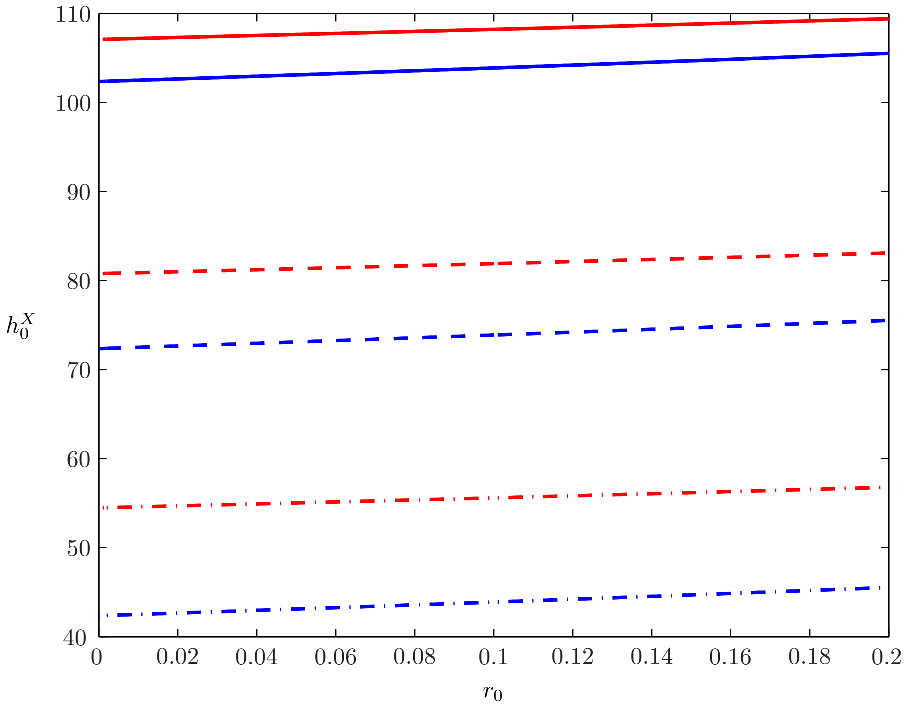
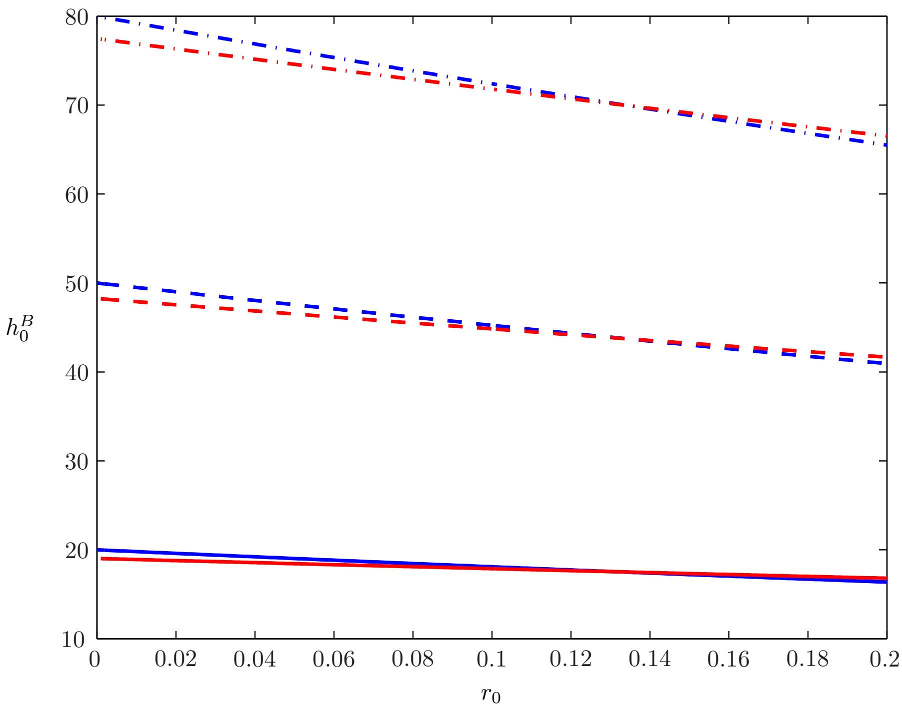


Figure 4 shows the hedge ratio of the locally risk-minimizing strategy as a function of the interest rate level. One can see that treating the interest rate, that is stochastic in reality, as a constant will decrease the number of zeros (with total loss in case of default) held in the hedging strategy below the optimal level, hence leading to a position less risky than necessary. The converse holds for the position in the money market account. From Figure 5 we can tell that this position is higher in the deterministic interest rates case.

Figure 5: *Initial position in the money market account as a function of the interest rate level.*

The figure shows the initial position in the money market account as a function of the interest rate level for an expected recovery payment of $\mu^Z = 20$ (solid lines), $\mu^Z = 50$ (dashed lines) and $\mu^Z = 80$ (dashed-dotted lines). The blue graphs illustrate the case of deterministic interest rates for parameters $t = 0$, $T = 1$, $\widehat{\lambda} = 2$, $C = 7$ and $F = 100$. The red graphs illustrate the case of stochastic interest rates with CIR dynamics for parameters $\kappa^r = 0.5$, $\theta^r = 0.05$ and $\sigma^r = 0.2$.

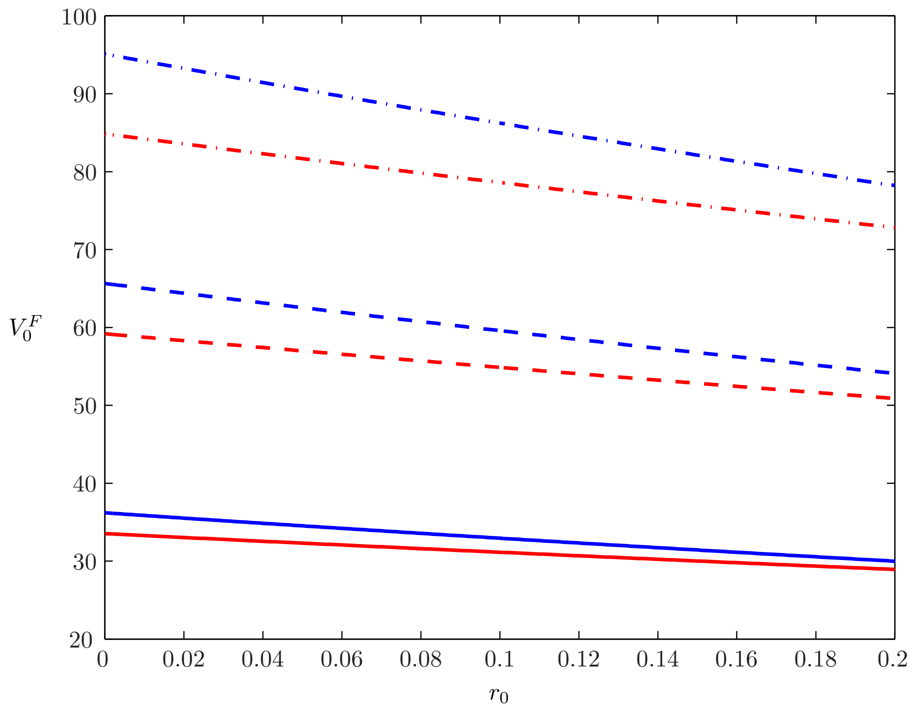


From Figure 6 we see that the second effect outweighs the first, i.e. the value of the credit derivative is overestimated in the deterministic interest rates model.

Therefore, modeling the interest rate as a stochastic process will reduce the hedging costs and thus improve the hedging quality.

Figure 6: *Initial portfolio value as a function of the interest rate level.*

The figure shows the initial portfolio value as a function of the interest rate level for an expected recovery payment of $\mu^Z = 20$ (solid lines), $\mu^Z = 50$ (dashed lines) and $\mu^Z = 80$ (dashed-dotted lines). The blue graphs illustrate the case of deterministic interest rates for parameters $t = 0$, $T = 1$, $\widehat{\lambda} = 2$, $C = 7$ and $F = 100$. The red graphs illustrate the case of stochastic interest rates with CIR dynamics for parameters $\kappa^r = 0.5$, $\theta^r = 0.05$ and $\sigma^r = 0.2$.



4.2 Stochastic Intensities

Suppose now that the default rate follows a stochastic process while the interest rate is a deterministic function. The $\widehat{P}$-dynamics of the discounted defaultable risky asset price are then given by[11]

$$dX_t = \widehat{\lambda}_t X_{t-} dt + \frac{1}{B_t} L^{\Lambda}_{t-} dm_t - X_{t-} dH_t,$$

where the processes L^{Λ} and m are given by

$$\begin{aligned} L^{\Lambda}_t &= \mathbb{1}_{\{\tau>t\}} \frac{1}{\widehat{P}(\tau > t|\mathcal{G}_t)} = \mathbb{1}_{\{\tau>t\}} \exp\{\Lambda_t\}, \\ m_t &= \widehat{E}\left[\frac{B_t}{B_T} \widehat{P}(\tau > t|\mathcal{G}_t) \big| \mathcal{G}_t\right] = \frac{B_t}{B_T} \widehat{E}\left[\exp\left\{-\Lambda_T\right\} \big| \mathcal{G}_t\right], \end{aligned}$$

and where Λ denotes the cumulative intensity process, i.e.

$$\Lambda_t = \int_0^t \widehat{\lambda}_s \, ds.$$

In particular, the conditional survival probability is given by

$$\widehat{P}(\tau > t|\mathcal{G}_t) = \exp\{-\Lambda_t\}.$$

and we have

$$d\langle X,X\rangle_t = \widehat{\lambda}_t \, X_{t-} dt.$$

For g^Z, g^F and g^C, we now have

$$g^Z_t = \frac{1}{B_T} \widehat{E}\left[\int_t^T \exp\left\{-\int_t^u \widehat{\lambda}_s ds\right\} \widehat{\lambda}_u \big| \mathcal{G}_t\right] \mu^Z(u) \, du, \tag{18}$$

$$g^F_t = \frac{1}{B_T} \widehat{E}\left[\exp\left\{-\int_t^T \widehat{\lambda}_s \, ds\right\} \big| \mathcal{G}_t\right] \cdot F, \tag{19}$$

$$g^C_t = \int_t^T \frac{1}{B_u} \widehat{E}\left[\exp\left\{-\int_t^u \widehat{\lambda}_s \, ds\right\} \big| \mathcal{G}_t\right] dC_u, \tag{20}$$

respectively.

[11] Using Proposition 2 in Blanchet-Scalliet and Jeanblanc (2004), this result is a direct consequence of the fact that r and $\widehat{\lambda}$ are independent.

Example 3 Suppose now that it is the intensity that follows the CIR model under the minimal martingale measure, i.e.

$$d\widehat{\lambda}_t = \kappa^{\widehat{\lambda}}(\theta^{\widehat{\lambda}} - \widehat{\lambda}_t)dt + \sigma^{\widehat{\lambda}}\sqrt{\widehat{\lambda}_t}d\widehat{W}_t,$$

where $\kappa^{\widehat{\lambda}}$, $\theta^{\widehat{\lambda}}$, $\sigma^{\widehat{\lambda}}$, $\widehat{\lambda}_0 > 0$. Thus

$$\widehat{E}\left[\exp\left\{-\int_0^T \widehat{\lambda}_s\, ds\right\} |\mathcal{F}_t\right] = \exp\left\{-\int_0^t \widehat{\lambda}_s\, ds - \widehat{\lambda}_t C(t,T) - D(t,T)\right\}, \qquad (21)$$

where $C(t,T)$ and $D(t,T)$ are given by (16) and (17) with σ^r replaced by $\sigma^{\widehat{\lambda}}$ and $\gamma^{\widehat{\lambda}} = \frac{1}{2}\sqrt{(\kappa^{\widehat{\lambda}})^2 + 2(\sigma^{\widehat{\lambda}})^2}$.

Defining $G_t = \widehat{P}(\tau > t|F_t)$ for all t, we have

$$\begin{aligned} m_t &= \frac{F}{B_T}\widehat{E}[G_s|\mathcal{F}_t] + \frac{1}{B_T}\int_0^T \widehat{E}[G_s\widehat{\lambda}_s|\mathcal{F}_t]\mu^Z(s)\, ds + \int_0^T \frac{1}{B_s}\widehat{E}[G_s|\mathcal{F}_t]\, dC_s \\ &=: u(t,r_t). \end{aligned}$$

From Brigo and Mercurio (2006, p. 822), we get

$$\begin{aligned} &\widehat{E}[G_s\widehat{\lambda}_s|\mathcal{F}_t] \\ =\ & -\frac{\partial}{\partial s}\widehat{E}[G_s|\mathcal{F}_t] \\ =\ & \widehat{E}[G_s|\mathcal{F}_t] \cdot \left[\left(1 - \kappa^{\widehat{\lambda}}C(t,s) + \frac{(\sigma^{\widehat{\lambda}})^2}{2}C^2(t,s)\right)\widehat{\lambda}_t + \kappa^{\widehat{\lambda}}\theta^{\widehat{\lambda}}C(t,s)\right] \qquad (22) \end{aligned}$$

From Proposition A.1, it follows that the process ξ is given by

$$\begin{aligned} \xi_t &= \sigma^r\sqrt{\widehat{\lambda}_t}\,\frac{\partial}{\partial\widehat{\lambda}}u(t,\widehat{\lambda}_t) \\ &= \sigma^r\sqrt{\widehat{\lambda}_t} \cdot \left[-C(t,T)\frac{F}{B_T}\widehat{E}[G_T|\mathcal{F}_t] + \frac{1}{B_T}\int_t^T \frac{\partial}{\partial\widehat{\lambda}}\widehat{E}[G_s\widehat{\lambda}_s|\mathcal{F}_t]\mu^Z(s)\, ds \right.\\ &\qquad \left. - \int_t^T C(t,s)\frac{1}{B_s}\widehat{E}[G_s|\mathcal{F}_t]\, dC_s\right] \end{aligned}$$

Since

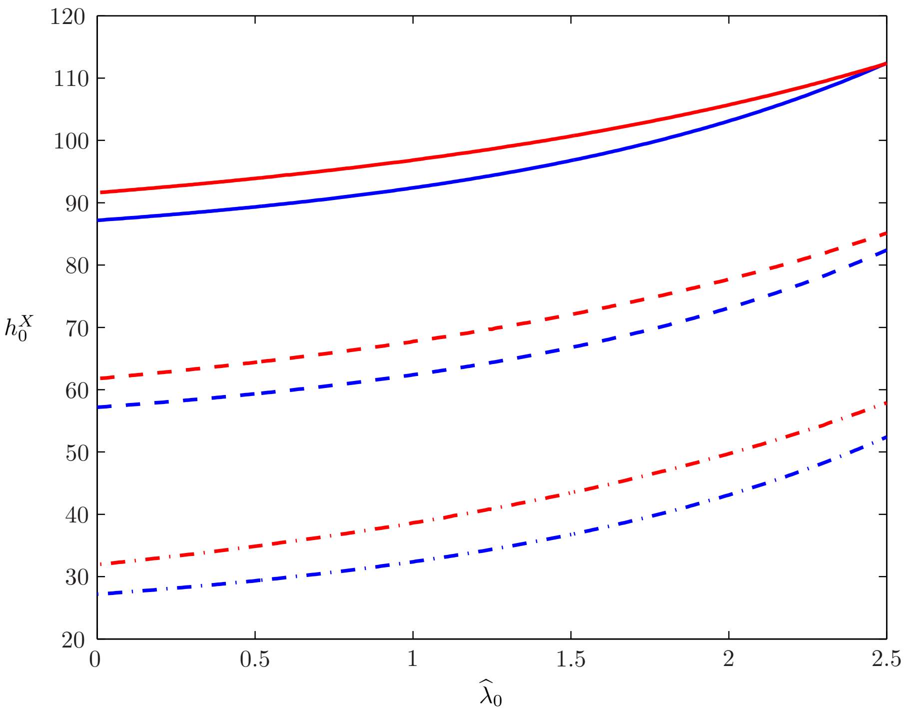
$$\begin{aligned} &\frac{\partial}{\partial\widehat{\lambda}}\widehat{E}[G_s\widehat{\lambda}_s|\mathcal{F}_t] \\ =\ & -C(t,s) \cdot \widehat{E}[G_s\widehat{\lambda}_s|\mathcal{F}_t] + \widehat{E}[G_s|\mathcal{F}_t] \cdot \left(1 - \kappa^{\widehat{\lambda}}C(t,s) + \frac{(\sigma^{\widehat{\lambda}})^2}{2}C^2(t,s)\right), \qquad (23) \end{aligned}$$

the hedging strategy is again given explicitly. □

Figure 7 shows the hedge ratio of the locally risk-minimizing strategy as a function of the intensity. One can see that the hedge ratio is an increasing function of the default rate, which might seem counterintuitive at first glance, since the higher the default rate, the higher the probability that the hedging instrument jumps to zero and becomes worthless. However, a higher intensity also means it is more likely that the investor (who is short in the credit derivative) will not have to pay the face value F but only the lower recovery payment Z.

Figure 7: *Initial hedge ratio as a function of the intensity.*

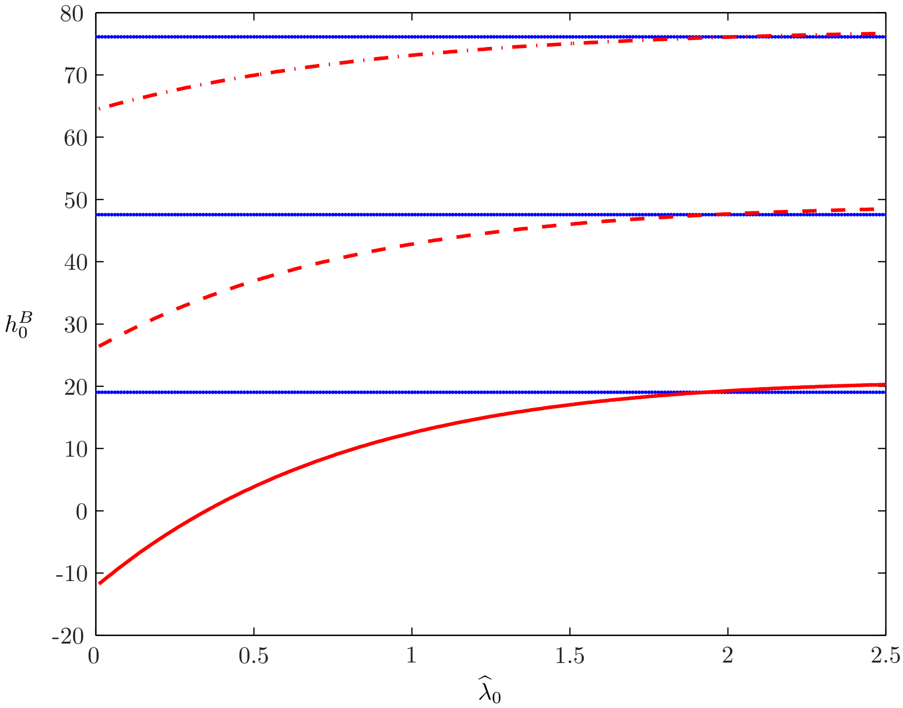
The figure shows the initial hedge ratio as a function of the default rate for an expected recovery payment of $\mu^Z = 20$ (solid lines), $\mu^Z = 50$ (dashed lines) and $\mu^Z = 80$ (dashed-dotted lines). The blue graphs illustrate the case of deterministic default rates for parameters $t = 0$, $T = 1$, $r = 0.05$, $C = 7$ and $F = 100$. The red graphs illustrate the case of stochastic default rates with CIR dynamics for parameters $\kappa^{\widehat{\lambda}} = 0.5$, $\theta^{\widehat{\lambda}} = 2$ and $\sigma^{\widehat{\lambda}} = 0.4$.



For the same reason the hedge ratio is decreasing in the expected recovery payment μ^Z. The higher the expected recovery payment, the more money has to be invested in the money market account for the investor to be able to pay it after the position in the zeros became worthless.

Figure 8: *Initial position in the money market account as a function of the intensity.*

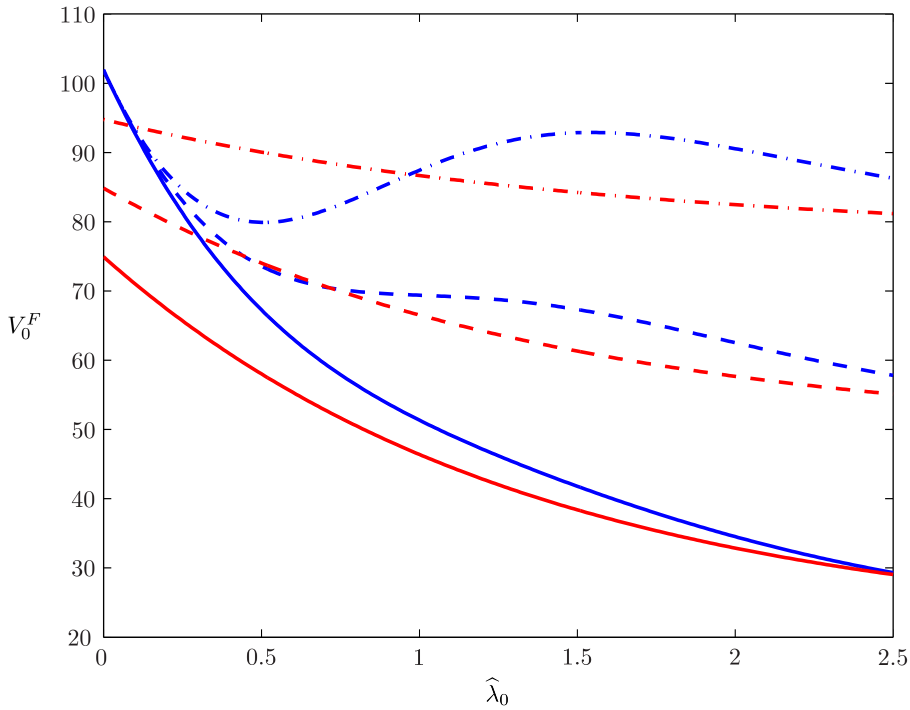
The figure shows the initial position in the money market account as a function of the default rate for an expected recovery payment of $\mu^Z = 20$ (solid lines), $\mu^Z = 50$ (dashed lines) and $\mu^Z = 80$ (dashed-dotted lines). The blue graphs illustrate the case of deterministic default rates for parameters $t = 0$, $T = 1$, $r = 0.05$, $C = 7$ and $F = 100$. The red graphs illustrate the case of stochastic default rates with CIR dynamics for parameters $\kappa^{\widehat{\lambda}} = 0.5$, $\theta^{\widehat{\lambda}} = 2$ and $\sigma^{\widehat{\lambda}} = 0.4$.



From Figure 7, one can see that modelling the default rate as a constant will generally (i.e. apart from the case of very low default rates) reduce the number of defaultable zeros held in the hedging portfolio below the optimal level. The position in the money market account, however is not affected by level of the intensity, see Figure 8. This is due to the fact that, at any time, the value of the position in the money market account equals the cumulative value of the credit derivative if default was to occur an instant from now. This is also why it is increasing in the expected recovery payment.

Figure 9: *Initial portfolio value as a function of the intensity.*

The figure shows the initial portfolio value as a function of the default rate for an expected recovery payment of $\mu^Z = 20$ (solid lines), $\mu^Z = 50$ (dashed lines) and $\mu^Z = 80$ (dashed-dotted lines). The blue graphs illustrate the case of deterministic default rates for parameters $t = 0$, $T = 1$, $r = 0.05$, $C = 7$ and $F = 100$. The red graphs illustrate the case of stochastic default rates with CIR dynamics for parameters $\kappa^{\widehat{\lambda}} = 0.5$, $\theta^{\widehat{\lambda}} = 2$ and $\sigma^{\widehat{\lambda}} = 0.4$.



From Figure 9, we can see that, for low expected recovery rates, the portfolio value in the deterministic and the stochastic intensity case lie close. For high expected recovery payments, however, the portfolio value, and hence the value of the credit derivate, is underestimated in the deterministic intensity model. Therefore, modeling the default rate as a stochastic process will, at least for low intensities, i.e. for rare events, reduce the hedging costs and thus improve the hedging quality.

4.3 Stochastic Interest Rates and Stochastic Intensities

We now assume that both the interest and the default rate follow a stochastic process and, in particular, that $\mathcal{F}_t = \sigma(W_t^r, W_t^{\widehat{\lambda}})$ for two independent[12] Brownian motions.

The defaultable zero bond price follows the dynamics

$$dX_t = (1 - H_t)\widehat{\lambda}_t X_{t-}\, dt + (1 - H_t)\, \frac{1}{B_t G_t}\, dm_t^X - X_{t-} dH_t,$$

where the process m^X is now given by

$$m_t^X = \widehat{E}\left[\frac{B_t}{B_T} G_T \middle| \mathcal{F}_t\right].$$

The martingale representations of the processes m^X and m now take the form

$$\begin{aligned} m_t &= m_0 + \int_0^t \xi_s^{m,r}\, dW^r + \int_0^t \xi_s^{m,\widehat{\lambda}}\, dW^{\widehat{\lambda}}, \\ m_t^X &= m_0 + \int_0^t \xi_s^{X,r}\, dW^r + \int_0^t \xi_s^{X,\widehat{\lambda}}\, dW^{\widehat{\lambda}}, \end{aligned}$$

for $(\mathcal{F}_t)$-predictable processes $\xi^{\cdot,\cdot}$.

Lemma 3 (FS-Decomposition, Case of a Two-Dimensional BM)
The discounted cumulative value V_T^F of the credit derivative (Z, C, F) at maturity has the following strong Föllmer-Schweizer-decomposition:

$$V_T^F = V_0^F + \int_0^T h_t^X\, dX_t + L_T^F\ ,$$

[12] Brigo and Mercurio (2006, p. 817) consider the case of two correlated Brownian motions with correlation coefficient ρ, i.e. $dW^r dW^{\widehat{\lambda}} = \rho dt$, and show that there exists no explicit representation of the zero bond price in case $\rho \neq 0$, but that the impact of ρ is negligible. Thus this independence assumption is no major restriction.

where

$$
\begin{aligned}
h_t^X &= \frac{d\langle V^F, X\rangle_t}{d\langle X\ , X\rangle_t} \\
&= (1-H_t)\left(B_t \cdot \frac{\xi_t^m}{\xi_t^X} + \frac{g_{t-}^C + g_{t-}^F + g_{t-}^Z}{X_{t-}} - \frac{\mu^Z(t)}{\widehat{E}[B_T|\mathcal{F}_t]X_{t-}}\right),
\end{aligned}
$$

is the locally risk-minimizing hedge ratio, $V_0^F = g_0^C + g_0^F + g_0^Z$ *is a constant,* L^F *is a martingale which is orthogonal to* M*, given by* $L_t^F = \int_0^t \frac{1}{B_T}(Z(s) - \mu^Z(s))\, d\widetilde{H}_s$*, and the processes* ξ^m *and* ξ^X *are given by*

$$
\begin{aligned}
\xi^m &= \xi_t^{m,r} + \xi_t^{m,\widehat{\lambda}}, \\
\xi^X &= \xi_t^{X,r} + \xi_t^{X,\widehat{\lambda}}.
\end{aligned}
$$

This yields the LRM-strategy in case both the interest rate and the intensity are stochastic.

Proposition 5 (LRM-Hedge, Case of a Two-Dimensional BM)
In case of stochastic interest rates, the locally risk-minimizing hedging strategy of the credit derivative (Z,C,F) *is given by*

$$
\begin{aligned}
h_t^X &= B_t \cdot \frac{\xi_t^m}{\xi_t^X} + \frac{g_{t-}^C + g_{t-}^F + g_{t-}^Z}{X_{t-}} - \frac{\mu^Z(t)}{\widehat{E}[B_T|\mathcal{F}_t]X_{t-}}, \\
h_t^B &= \int_0^t \frac{1}{B_s}\, dC_s + \frac{\mu^Z(t)}{\widehat{E}[B_T|\mathcal{F}_t]X_{t-}} - B_t \cdot \frac{\xi_t^m}{\xi_t^X} \cdot X_t,
\end{aligned}
$$

for $t \le \tau$*, and*

$$
\begin{aligned}
h_t^X &= 0, \\
h_t^B &= \int_0^\tau \frac{1}{B_s}\, dC_s + \widehat{E}\left[\frac{1}{B_T}|\mathcal{F}_t\right] Z_\tau,
\end{aligned}
$$

for $t > \tau$*.*

Example 3 Suppose both the interest rate and the intensity follow a CIR-process, i.e.

$$
\begin{aligned}
dr_t &= \kappa^r(\theta^r - r_t)dt + \sigma^r \sqrt{r_t} d\widehat{W}_t^r, \\
d\widehat{\lambda}_t &= \kappa^{\widehat{\lambda}}(\theta^{\widehat{\lambda}} - \widehat{\lambda}_t)dt + \sigma^{\widehat{\lambda}} \sqrt{\widehat{\lambda}_t} d\widehat{W}_t^{\widehat{\lambda}}.
\end{aligned}
$$

The process m can be written

$$\begin{aligned} m_t &= \widehat{E}\left[\frac{G_T}{B_T}F + \int_0^T \frac{G_s}{B_T} Z_s \widehat{\lambda}_s ds + \int_0^T \frac{G_s}{B_s} dC_s \big| \mathcal{F}_t\right] \\ &= F \cdot \widehat{E}\left[\frac{1}{B_T}\big|\mathcal{F}_t\right] \widehat{E}\left[G_T \big| \mathcal{F}_t\right] + \widehat{E}\left[\frac{1}{B_T}\big|\mathcal{F}_t\right] \int_0^T \widehat{E}\left[G_s \widehat{\lambda}_s \big| \mathcal{F}_t\right] \mu^Z(s)\, ds \\ &\qquad + \int_0^T \widehat{E}\left[\frac{1}{B_s}\big|\mathcal{F}_t\right] \widehat{E}\left[G_s \big| \mathcal{F}_t\right] dC_s \\ &=: u(t,r_t,\widehat{\lambda}_t). \end{aligned}$$

From Proposition A.2, it follows that the processes $\xi^{m,r}$ and $\xi^{m,\widehat{\lambda}}$ are given by

$$\begin{aligned} \xi_t^{m,r} &= \sigma^r \sqrt{r_t} \cdot \frac{\partial}{\partial r} u(t,r,\widehat{\lambda}), \\ \xi_t^{m,\widehat{\lambda}} &= \sigma^{\widehat{\lambda}} \sqrt{\widehat{\lambda}_t} \cdot \frac{\partial}{\partial \widehat{\lambda}} u(t,r,\widehat{\lambda}). \end{aligned}$$

Since

$$\begin{aligned} &\frac{\partial}{\partial r} u(t,r,\widehat{\lambda}) \\ = &-C^r(t,T)\, \widehat{E}\left[\frac{1}{B_T}\big|\mathcal{F}_t\right] \cdot \left(F \cdot \widehat{E}\left[G_T \big| \mathcal{F}_t\right] + \int_0^T \widehat{E}\left[G_s \widehat{\lambda}_s \big| \mathcal{F}_t\right] \mu^Z(s)\, ds\right) \\ &\qquad - \int_t^T C^r(t,s)\, \widehat{E}\left[\frac{1}{B_s}\big|\mathcal{F}_t\right] \widehat{E}\left[G_s \big| \mathcal{F}_t\right] dC_s \end{aligned}$$

and

$$\begin{aligned} \frac{\partial}{\partial \widehat{\lambda}} u(t,r,\widehat{\lambda}) &= -C^{\widehat{\lambda}}(t,T) \cdot F \cdot \widehat{E}\left[\frac{1}{B_T}\big|\mathcal{F}_t\right] \widehat{E}\left[G_T \big| \mathcal{F}_t\right] \\ &\qquad + \widehat{E}\left[\frac{1}{B_T}\big|\mathcal{F}_t\right] \int_0^T \frac{\partial}{\partial \widehat{\lambda}} \widehat{E}\left[G_s \widehat{\lambda}_s \big| \mathcal{F}_t\right] \mu^Z(s)\, ds \\ &\qquad - \int_t^T C^{\widehat{\lambda}}(t,s)\, \widehat{E}\left[\frac{1}{B_s}\big|\mathcal{F}_t\right] \widehat{E}\left[G_s \big| \mathcal{F}_t\right] dC_s, \end{aligned}$$

we have

$$
\begin{aligned}
\xi_t^{m,r} &= \sigma^r \sqrt{r_t} \cdot \Bigg[-C^r(t,T)\, \widehat{E}\left[\frac{1}{B_T}\Big|\mathcal{F}_t\right] \\
&\qquad \cdot \left(F \cdot \widehat{E}\left[G_T \big| \mathcal{F}_t\right] + \int_0^T \widehat{E}\left[G_s \widehat{\lambda}_s \big| \mathcal{F}_t\right] \mu^Z(s)\, ds \right) \\
&\qquad - \int_t^T C^r(t,s)\, \widehat{E}\left[\frac{1}{B_s}\Big|\mathcal{F}_t\right] \widehat{E}\left[G_s|\mathcal{F}_t\right] dC_s \Bigg], \qquad (24) \\
\xi_t^{m,\widehat{\lambda}} &= \sigma^{\widehat{\lambda}} \sqrt{\widehat{\lambda}_t} \cdot \Bigg[-C^{\widehat{\lambda}}(t,T) \cdot F \cdot \widehat{E}\left[\frac{1}{B_T}\Big|\mathcal{F}_t\right] \widehat{E}\left[G_T|\mathcal{F}_t\right] \\
&\qquad + \widehat{E}\left[\frac{1}{B_T}\Big|\mathcal{F}_t\right] \int_0^T \frac{\partial}{\partial \widehat{\lambda}} \widehat{E}\left[G_s \widehat{\lambda}_s \big| \mathcal{F}_t\right] \mu^Z(s)\, ds \\
&\qquad - \int_t^T C^{\widehat{\lambda}}(t,s)\, \widehat{E}\left[\frac{1}{B_s}\Big|\mathcal{F}_t\right] \widehat{E}\left[G_s|\mathcal{F}_t\right] dC_s \Bigg], \qquad (25)
\end{aligned}
$$

Since the process m^X can be written

$$
\begin{aligned}
m_t^X &= B_t\, \widehat{E}\left[G_T|\mathcal{F}_t\right] \widehat{E}\left[\frac{1}{B_T}\Big|\mathcal{F}_t\right] \\
&=: v(t,r_t,\widehat{\lambda}_t),
\end{aligned}
$$

it follows from Theorem 15.4.1 in Bruti-Liberati and Platen (2010) that the processes $\xi^{X,r}$ and $\xi^{X,\widehat{\lambda}}$ are given by

$$
\begin{aligned}
\xi_t^{X,r} &= \sigma^r \sqrt{r_t} \cdot \frac{\partial}{\partial r} v(t,r,\widehat{\lambda}), \\
\xi_t^{X,\widehat{\lambda}} &= \sigma^{\widehat{\lambda}} \sqrt{\widehat{\lambda}_t} \cdot \frac{\partial}{\partial \widehat{\lambda}} v(t,r,\widehat{\lambda}).
\end{aligned}
$$

Since

$$
\begin{aligned}
\frac{\partial}{\partial r} v(t,r,\widehat{\lambda}) &= -B_t\, C^r(t,T)\, \widehat{E}\left[\frac{1}{B_T}\Big|\mathcal{F}_t\right] \widehat{E}\left[G_T|\mathcal{F}_t\right], \\
\frac{\partial}{\partial \widehat{\lambda}} v(t,r,\widehat{\lambda}) &= -B_t\, C^{\widehat{\lambda}}(t,T)\, \widehat{E}\left[\frac{1}{B_T}\Big|\mathcal{F}_t\right] \widehat{E}\left[G_T|\mathcal{F}_t\right],
\end{aligned}
$$

we get

$$
\begin{aligned}
\xi_t^{X,r} &= -\sigma^r \sqrt{r_t}\, C^r(t,T)\, B_t\, \widehat{E}\left[\frac{1}{B_T}\Big|\mathcal{F}_t\right] \widehat{E}\left[G_T|\mathcal{F}_t\right], \qquad (26) \\
\xi_t^{X,\widehat{\lambda}} &= -\sigma^{\widehat{\lambda}} \sqrt{\widehat{\lambda}_t}\, C^{\widehat{\lambda}}(t,T)\, B_t\, \widehat{E}\left[\frac{1}{B_T}\Big|\mathcal{F}_t\right] \widehat{E}\left[G_T|\mathcal{F}_t\right]. \qquad (27)
\end{aligned}
$$

Since one can plug (15), (21), (22) and (23) into (24)-(27), the hedging strategy is again given explicitly. □

5 Simulation of Hedging Costs

In this section, we run a simulation with 10,000 iterations to test the impact of the different model assumptions on the cumulative hedging costs. We also test the LRM-strategy against strategies using alternative hedging instruments such as CDS contracts, CoCo-bonds, a defaultable zero coupon bond that trades at a stochastic spread in the default intensity relative to the credit derivative we wish to hedge, and a credit index.

If not specified otherwise, we use the following parameters: The credit derivative (C, F, Z) is assumed to pay an annualized coupon at rate $c = 0.08$ and to have a promised payment of $F = 100$. The doubly-stochastic fraction of this payment recoverd in case of default is assumed to have a Beta $(12, 12)$-distribution, i.e. $\mu^Z(t) = 50$ for all t. We assume a maturity of $T = 2$ years and that the hedging strategies are adjusted on a weekly basis, i.e. we consider the trading dates $t_0 = 0 < t_1 < ... < t_n = 2$ with $t_i - t_{i-1} = 1/52$ for all $i = 1,...,104$.

For the case of both deterministic interest and default rate, we use constant rates of $r = 0.05$ and $\widehat{\lambda} = 0.35$. To simulate the CIR-model for the stochastic interest respectively default rate, we proceed as described by Glasserman (2003, p. 120ff.). As is mentioned there, a simple Euler discretization of the form

$$\begin{aligned} r(t_{i+1}) &= r(t_i) + \kappa^r(\theta^r - r(t_i)) \cdot (t_{i+1} - t_i) + \sigma^r \sqrt{r(t_i)(t_{i+1} - t_i)} Z^r_{i+1} \\ \widehat{\lambda}(t_{i+1}) &= \widehat{\lambda}(t_i) + \kappa^{\widehat{\lambda}}(\theta^{\widehat{\lambda}} - \widehat{\lambda}(t_i)) \cdot (t_{i+1} - t_i) + \sigma^{\widehat{\lambda}} \sqrt{\widehat{\lambda}(t_i)(t_{i+1} - t_i)} Z^{\widehat{\lambda}}_{i+1}, \end{aligned}$$

where $Z^r_1,...,Z^r_n$ and $Z^{\widehat{\lambda}}_1,...,Z^{\widehat{\lambda}}_n$ are independent standard normal random variables, will still produce negative values, even if the expressions under the square root are replaced by their positive parts. We therefore use the algorithm from Glasserman (2003, p. 124) that allows to sample from the exact transition law of the processes. The respective parameters are given by $\theta^r = 0.05$, $\kappa^r = 0.01$, $\sigma^r = 0.01$ and $r_0 = 0.05$ for the interest rate and $\theta^{\widehat{\lambda}} = 0.35$, $\kappa^{\widehat{\lambda}} = 0.25$, $\sigma^{\widehat{\lambda}} = 0.4$ and $\widehat{\lambda}_0 = 0.35$ for the default rate. We first examine the basic model with both deterministic interest and default rate. From Table 2 we see that the hedger, on average, faces nearly zero additional costs apart from the initial investment in the amount of the initial value V^F_0 of the credit derivative (C, F, Z) to set up the strategy, i.e. the strategy is mean-self-financing. Additional costs accrue if default occurs before maturity and the doubly-stochastic recovery payment deviates from its expected value of $\mu^Z(t) = 50$. For instance, the highest cumulative hedging costs of 111.32 in the simulation are

Table 2: Discounted cumulative hedging costs when hedging a defaultable coupon-paying bond.

	Hedging Instruments						
Total Costs		Junior	Bond		Stock	CDS	Junior Bond
	$r, \widehat{\lambda}$ determ.	r stoch.	$\widehat{\lambda}$ stoch.	$r \& \widehat{\lambda}$ stoch.			stoch. spread
(Initial Costs)	(78.68)	(78.68)	(80.13)	(79.89)	(78.68)	(78.68)	(78.68)
Mean	**78.79**	**78.79**	**79.94**	**79.94**	**80.34**	**79.75**	**77.07**
Std Dev	6.44	6.47	6.52	6.48	6.67	1.72	16.35
Skewness	0.15	0.37	0.62	0.55	-0.34	-1.42	-0.62
Kurtosis	5.37	5.36	5.37	5.43	4.99	10.12	3.32
Min	52.32	52.78	53.74	53.41	50.15	64.96	20.72
Max	111.32	112.73	113.30	113.93	110.57	90.14	106.91
99%-quantile	97.70	98.15	99.89	99.78	98.18	84.25	105.07
95%-quantile	90.73	91.15	92.88	92.56	91.39	81.87	102.53
90%-quantile	86.89	87.31	88.84	88.65	87.14	80.85	98.95
75%-quantile	78.89	79.40	81.36	81.38	82.87	80.13	86.86
50%-quantile	78.68	78.23	78.89	79.06	81.13	80.13	78.48
25%-quantile	78.68	77.76	77.52	77.66	78.57	79.76	69.28
10%-quantile	70.82	71.33	73.58	73.35	71.19	77.73	53.17
5%-quantile	67.00	67.46	69.39	69.17	67.37	76.27	44.04
1%-quantile	60.83	61.47	63.32	63.06	61.08	73.40	33.05

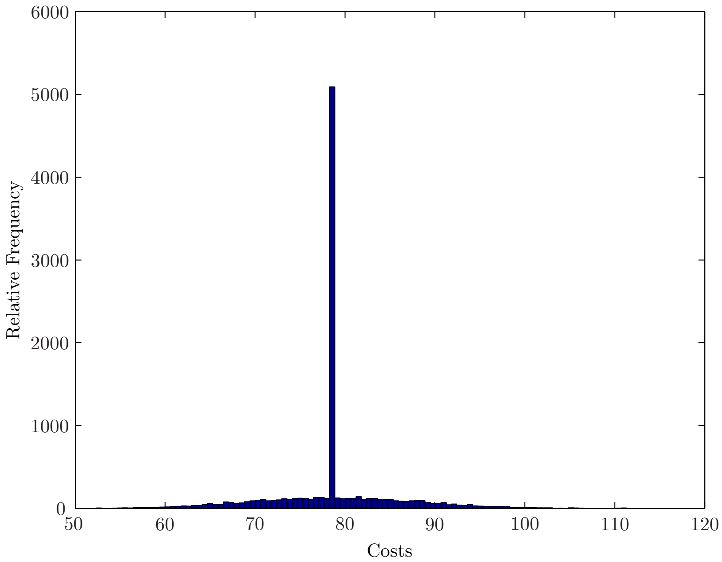
due to realized recovery payment of 85.92. In this case, prior to default the position h_t^B in the money market account from Proposition 2 is far too low. Conversely, the lowest cumulative hedging costs of 52.32 in the simulation correspond to a realized recovery payment of only 20.75. In this case, the position in the money market account was far too high. If no default occurs prior to maturity, the hedging strategy reduces to the replication strategy in case of a single-stochastic recovery payment. Hence the strategy is self-financing and no additional costs accrue, i.e. the cumulative costs equal the initial costs of 78.68. In case only the interest rate is stochastic, the hedging costs remain nearly unaffected, cf. Table 2, since the interest rate risk is perfectly hedgeable, and the small differences in the hedging costs are thus solely due to the discretization error. In contrast, if the default rate is stochastic, the hedging costs are affected, especially in iterations where no default occurs, i.e. when the hedging strategy is adjusted for variation in the intensity at any trading date, but this this turns out to have been unnecessary since there is no default. Moreover, the simulation results show that the hedging costs of the LRM-strategy are only slightly higher in the practically more relevant case of using stocks as the hedging instrument.

Let us now consider the alternative strategies. First, we consider the duplication strategy using CDS contracts by Bielecki, Jeanblanc and Rutkowski (2007). This strategy involves a short position in CDS contracts and a long position in the money market account. In a continuous-time setting, it duplicates the claim to hedge, and it can be seen from Table 2 that the hedging costs due to the discretization are very small. The variance is smaller than for all other strategies considered and both the minimum and the maximum costs are much closer to the expected costs, see also part (a) of Figure 11. However, as mentioned by Bielecki, Jeanblanc and Rutkowski

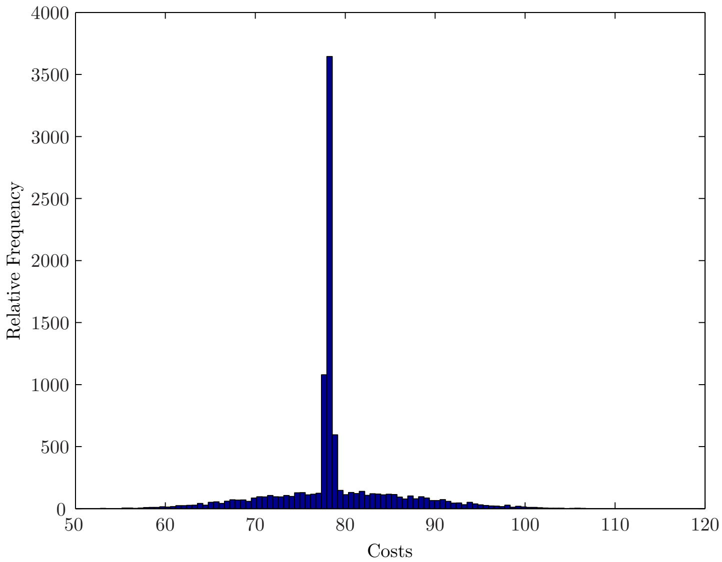
(2008, p. 2512f.), the strategy involves trading a CDS contract issued in the past, i.e. an instrument that is not very liquid in practice.

Figure 10: *Discounted cumulative hedging costs of LRM-strategy with defaultable zero bond.*

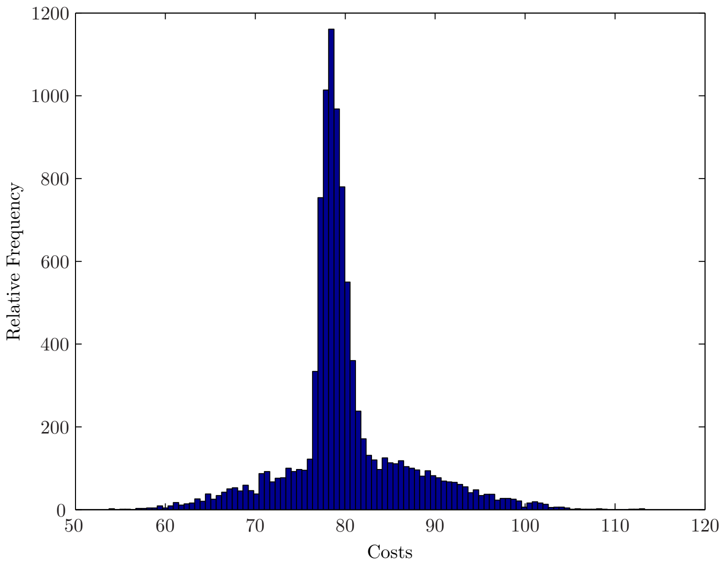
(a) Deterministic interest and default rate.



(b) Stochastic interest rate and deterministic default rate.



(c) Deterministic interest rate and stochastic default rate.



(d) Stochastic interest rate and default rate.

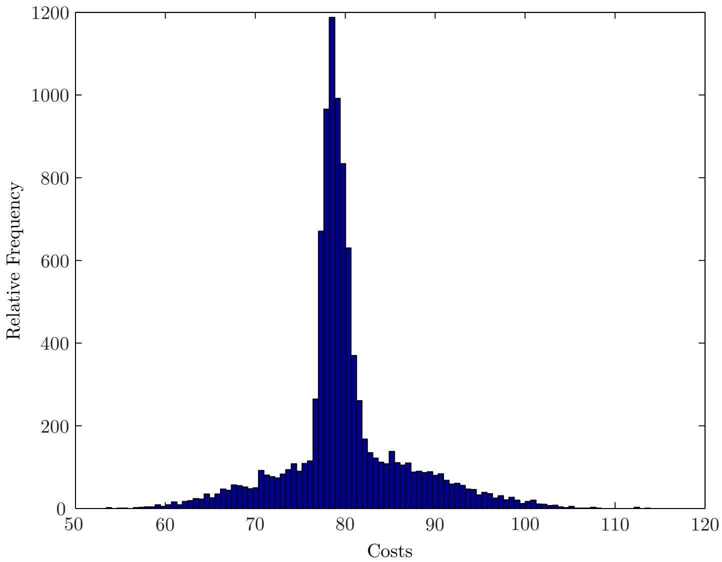
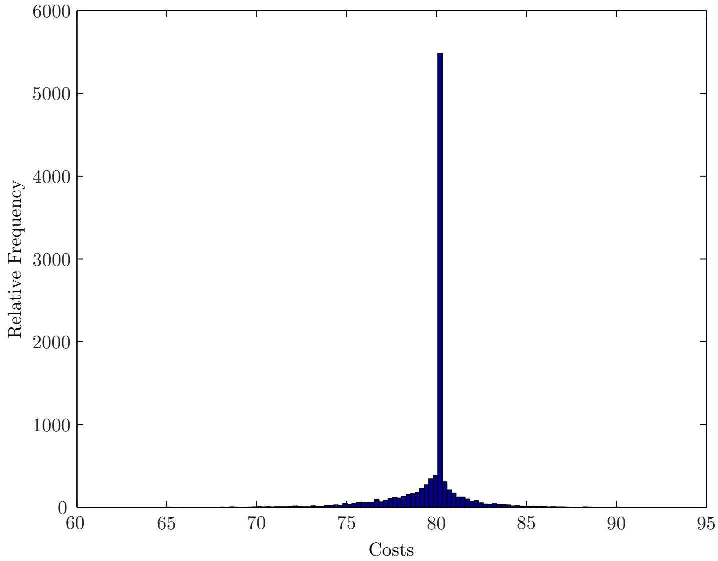
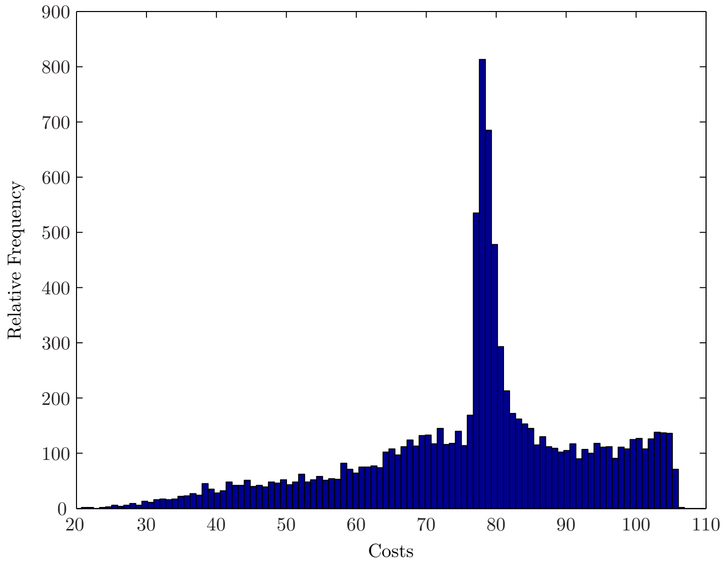


Figure 11: *Discounted cumulative hedging costs for alternative hedging instruments.*

(a) CDS.



(b) Stochastic Spread.



Finally, we also consider two cross-hedging strategies. The first of them involves a defaultable zero coupon bond that trades at a spread (in the default intensity) relative to the credit derivative we wish to hedge. In particular, the default times of the hedging instrument and the claim to hedge are independent in this case. The stochastic spread is modelled as a CIR-process with mean 0.05, i.e. the default probability of the hedging instrument is, on average, 5% higher (or lower) than the default intensity of the credit derivative (Z, C, F). Surprisingly, the sign of the spread turned out to be irrelevant with regard to the discounted cumulative hedging costs. More precisely, regardless of whether the spread is positive (as was the case for the simulation shown in Table 2) or negative, the discounted cumulative hedging costs will be smaller (77.09 compared to between 79 and 80 in the present case). This is due to a decreasing position in the even riskier hedging instrument in case of a positive spread, and due to the lower default probability of the hedging instrument in case the spread is negative.

6 Conclusion

There is overwhelming empirical evidence that recovery payments in case of default do not only depend on time of default and the term structure but also on additional sources of risk. Based on the concept of single-stochastic and doubly-stochastic recovery payments introduced in this paper, we derive hedging strategies which are locally risk minimizing (LRM). We denote the recovery rate as *single-stochastic* if the recovery amount depends only on the default event and the interest rate. We denote the recovery rate as *doubly-stochastic* if the recovery amount also depends on the realization of another random variable. Corresponding model variants are examined for the reduced-form model framework.

It turns out that the corresponding LRM-strategy is not only mean-self-financing but also self-financing if the default recovery is single-stochastic. That is, as long as the recovery amount is known in the event of default, there exists a self-financing replication strategy for credit derivatives. Moreover, we find that in the more realistic case of doubly-stochastic default recoveries, the LRM-hedging strategy does only depend on the *expected* recovery amount, not on other characteristics of its distribution. This key result of the paper helps to justify the simplifying assumption frequently made when valuing and hedging credit derivatives, that the default recovery is constant, conditional on the default event.

The key result also holds when replacing the zero coupon with total loss in case of default by another hedging instrument. For instance, under the assumption that the stock price jumps to/or reaches a pre-specified value when the credit event occurs, one may also use common stocks. Moreover, and in contrast to the existing literature, we derive explicit solutions for the hedge ratio even when all relevant quantities are stochastic. In our simulations, it turns out that it is crucial to model the default intensity as a stochastic process (in addition to a doubly-stochastic recovery payment).

Our key insight still remains valid when replacing the LRM-concept by another hedging concept which is based on a quadratic criterion. Moreover, it can be shown that the key message also holds when dealing with structural models.

A Appendix

Problem 2 (LRM-Hedge in continuous time)
A trading strategy $\mathbf{H}$ *with* $V_T(\mathbf{H}) = F_T$ P*-a.s. is called* locally risk-minimizing, *(LRM) for short, if it fulfills*

$$\liminf_{N\to\infty} r^{\mathcal{T}_N}(\mathbf{H},\Delta) \geq 0 \quad P_M\text{-a.s.}^{13}$$

for every null-convergent sequence of partitions $\mathcal{T}_N = \{t_0 = 0, t_1, \ldots, t_N = T\}$ *of* $[0,T]$, *i.e.* $\mathcal{T}_N \subset \mathcal{T}_{N+1}$ *and* $\lim_{N\to\infty} \max_{i=1,\ldots,N}(t_i^N - t_{i-1}^N) = 0$, *and every disturbance* Δ. *Here a* disturbance $\Delta = (\delta,\varepsilon)$ *is a trading strategy, such that* $\delta_T = \varepsilon_T = 0$ *and* $\int_0^T |\delta_s|\, d|A|_s$ *is bounded. Furthermore, defining the* remaining risk $R_t(\mathbf{H})$ *measured as the expected quadratic increase of the discounted hedging costs,* $R_t(\mathbf{H}) = E^P\left[(C_T(\mathbf{H}) - C_t(\mathbf{H}))^2 \,|\mathcal{G}_t\right]$, *the expression*

$$r^{\mathcal{T}}(\mathbf{H},\Delta) = \sum_{i=0}^{n-1} \frac{R_{t_i}(\mathbf{H} + \Delta|_{(t_i,t_{i+1}]}) - R_{t_i}(\mathbf{H})}{E^P[\langle M\rangle_{t_{i+1}} - \langle M\rangle_{t_i}|\mathcal{G}_{t_i}]} \mathbb{1}_{(t_i,t_{i+1}]}$$

denotes the risk quotient *for a trading strategy* $\mathbf{H}$, *a disturbance* $\Delta = (\delta,\varepsilon)$ *and the partition* $\mathcal{T} = \{t_0 = 0, t_1, \ldots, t_n = T\}$.

Hence, a trading strategy is locally risk-minimizing if a disturbance of the strategy will increase the risk measured by the risk quotient.

Proof of Lemma 1.
In proving this result,Theorem 2.4 from Schweizer (1991) applies directly, provided that the regularity conditions X(1) - X(5) are satisfied. From these five conditions[14,15] only X(2) is not satisfied since $\langle M\rangle_t = 0$ for $t > \tau$. Nonetheless, we can apply the results from Schweizer (1991) since for $t > \tau$ the financial market is not subject to any risk. In addition, $X_t = 0$ for $t > \tau$.

[13] $P_M = P \times \langle M,M\rangle$ denotes the Doléans Dade measure of $\langle M,M\rangle$ on the product space $\Omega \times [0,T]$ with the predictable σ-algebra.

[14] X(4) is satisfied if the default intensities fulfill the requirement $\mathrm{E}_M[|\widetilde{\alpha}| \log^+(|\widetilde{\alpha}|)] < \infty$. Here $\mathrm{E}_M[\cdot]$ denotes the expectation under the Doléans-Dade measure $P_M = P \times \langle M,M\rangle$. If the default intensities fulfill for example the condition $\inf_{t\in[0,T]} |\widehat{\lambda}(t) - \lambda(t)|/|\lambda(t)| > 0$, the requirement $\mathrm{E}_M[|\widetilde{\alpha}| \log^+(|\widetilde{\alpha}|)] < \infty$ holds.

[15] X(5) is satisfied since $P(\tau = T) = 0$ and X_T is P-a.s. continuous at T.

The following proof consists of two steps. In the first step we derive the locally risk-minimizing hedge ratio h_t^X, and in the second step we verify that L^F is a square-integrable martingale which is orthogonal to M. We have

$$\begin{aligned}
V_t^F &= \widehat{E}[\frac{F_T}{B_T}|\mathcal{G}_t] \\
&= \widehat{E}\left[\left(\int_0^T \frac{C_s}{B_s}\,ds + \frac{F}{B_T}\right)\mathbb{1}_{\{\tau>T\}} + \left(\int_0^\tau \frac{C_s}{B_s}\,ds + \frac{Z(\tau)}{B_T}\right)\mathbb{1}_{\{\tau\le T\}}\big|\mathcal{G}_t\right] \\
&= \mathbb{1}_{\{\tau\le t\}}\left(\int_0^\tau \frac{C_s}{B_s}\,ds + \frac{Z(\tau)}{\widehat{E}[B_T|F_t]}\right) + \mathbb{1}_{\{\tau>t\}}\int_0^t \frac{C_s}{B_s}\,ds \\
&\quad + \mathbb{1}_{\{\tau>t\}}\widehat{E}\left[\int_t^{T\wedge\tau}\frac{C_s}{B_s}\,ds + \mathbb{1}_{\{\tau>T\}}\frac{F}{B_T} + \mathbb{1}_{\{\tau\le T\}}\frac{Z(\tau)}{B_T}\big|\mathcal{G}_t\right] \\
&= H_t^Z + \mathbb{1}_{\{\tau>t\}}\int_0^t \frac{C_s}{B_s}\,ds \\
&\quad + \mathbb{1}_{\{\tau>t\}}\int_t^T \frac{C_u}{B_u}\exp\left\{-\int_t^u \widehat{\lambda}(s)\,ds\right\}\,du \\
&\quad + \mathbb{1}_{\{\tau>t\}}\frac{1}{B_T}\,\exp\left\{-\int_t^T \widehat{\lambda}(s)\,ds\right\}\cdot F \\
&\quad + \mathbb{1}_{\{\tau>t\}}\frac{1}{B_T}\int_t^T \exp\left\{-\int_t^u \widehat{\lambda}(s)ds\right\}\widehat{\lambda}(u)\,\mu^Z(u)\,du \\
&= H_t^Z + (1-H_t)\left(\int_0^t \frac{C_s}{B_s}\,ds + g_t^C + g_t^F + g_t^Z\right) \qquad \text{(A1)}
\end{aligned}$$

where

$$H_t^Z = \mathbb{1}_{\{\tau\le t\}}\left(\int_0^\tau \frac{C_s}{B_s}ds + \frac{Z(\tau)}{B_T}\right). \qquad \text{(A2)}$$

Thus

$$\begin{aligned}
\langle V^F, X\rangle_t &= \langle H^Z, X\rangle_t + \left\langle (1-H)\int_0^\cdot \frac{1}{B_s}ds, X\right\rangle_t \qquad \text{(A3)} \\
&\quad + \left\langle (1-H)g^C, X\right\rangle_t + \left\langle (1-H)g^F, X\right\rangle_t + \left\langle (1-H)g^Z, X\right\rangle_t .
\end{aligned}$$

For the first term on the right-hand side of (A3),

$$\begin{aligned}
[H^Z, X]_t &= H_t^Z X_t - \int_0^t H_{s-}^Z dX_s - \int_0^t X_{s-} dH_s^Z \\
&= 0 - 0 - \mathbb{1}_{\{\tau\le t\}} X_{\tau-}\cdot\Delta H_\tau^Z = -\mathbb{1}_{\{\tau\le t\}} X_{\tau-}\cdot H_\tau^Z
\end{aligned}$$

implies

$$
\begin{aligned}
d\langle H^Z,X\rangle_t &= \widehat{E}\left[d[H^Z,X]_t|\mathcal{G}_{t-}\right] \\
&= -\widehat{\lambda}(t)X_{t-}\left(\int_0^t \frac{C_s}{B_s}ds + \frac{\mu^Z(t)}{\widehat{E}[B_T|F_t]}\right)dt \\
&= \frac{-\int_0^t \frac{C_s}{B_s}ds - \frac{\mu^Z(t)}{\widehat{E}[B_T|F_t]}}{X_{t-}}\, d\langle X,X\rangle_t.
\end{aligned}
$$

Similarly, for the second term we get

$$
\begin{aligned}
d\left\langle (1-H)\int_0^{\cdot} \frac{C_s}{B_s}ds,X\right\rangle_t &= -d\left\langle H\int_0^{\cdot} \frac{C_s}{B_s}ds,X\right\rangle_t \\
&= \frac{\int_0^t \frac{C_s}{B_s}ds}{X_{t-}}\, d\langle X,X\rangle_t,
\end{aligned}
$$

while for the remaining terms, we have

$$
\begin{aligned}
d\left\langle (1-H)g^C,X\right\rangle_t &= \frac{g^C_{t-}}{X_{t-}}\, d\langle X,X\rangle_t, \\
d\left\langle (1-H)g^F,X\right\rangle_t &= \frac{g^F_{t-}}{X_{t-}}\, d\langle X,X\rangle_t, \\
d\left\langle (1-H)g^Z,X\right\rangle_t &= \frac{g^Z_{t-}}{X_{t-}}\, d\langle X,X\rangle_t,
\end{aligned}
$$

respectively. Altogether, by (A3),

$$
d\langle V^F,X\rangle_t = \left(-\frac{\mu^Z(t)}{X_{t-}B_T} + \frac{g^C_{t-}}{X_{t-}} + \frac{g^F_{t-}}{X_{t-}} + \frac{g^Z_{t-}}{X_{t-}}\right)\, d\langle X,X\rangle_t,
$$

so the locally risk-minimizing hedge ratio is given by

$$
h^X_t = -\frac{\mu^Z(t)}{X_{t-}B_T} + \frac{g^C_{t-}}{X_{t-}} + \frac{g^F_{t-}}{X_{t-}} + \frac{g^Z_{t-}}{X_{t-}}
$$

Since $H_0 = 0$ and $H^Z_0 = 0$,

$$
F_0 = \widehat{E}\left[\frac{F_T}{B_T}|\mathcal{G}_0\right] = g^C_0 + g^F_0 + g^Z_0, \tag{A4}
$$

by equation (A1).

In the following we verify that L^F is a square-integrable martingale with $L_0^F = 0$ which is P-orthogonal to M.

Since $L_0 = 0$, $\sup_{u\in[0,T]} \sigma^Z(u) < \infty$ by assumption and

$$\begin{aligned}
& \mathrm{E}[L_s|\mathcal{G}_t] \\
= \; & \mathrm{E}\left[\mathbb{1}_{\{\tau\le s\}}\frac{1}{B_T}\left(Z_\tau - \mu^Z(\tau)\right)|\mathcal{G}_t\right] \\
= \; & \mathrm{E}\left[\left(\mathbb{1}_{\{\tau\le t\}} + \mathbb{1}_{\{t<\tau\le s\}}\right)\frac{1}{B_T}\left(Z_\tau - \mu^Z(\tau)\right)|\mathcal{G}_t\right] \\
= \; & \mathbb{1}_{\{\tau<t\}}\frac{1}{B_T}\left(Z_\tau - \mu^Z(\tau)\right) \\
& + \int_t^s \frac{1}{B_T}\exp\left\{-\int_t^u \widehat{\lambda}(v)dv\right\}\widehat{\lambda}(u)\left(\mu^Z(u) - \mu^Z(u)\right)\, du \\
= \; & \mathbb{1}_{\{\tau\le t\}}\frac{1}{B_T}\left(Z_\tau - \mu^Z(\tau)\right) + 0 \\
= \; & L_t,
\end{aligned}$$

for $s \ge t$, L is a $(\mathcal{G})$-martingale.

L is stronly P-orthogonal to M since

$$\begin{aligned}
& \mathrm{E}[L_s M_s|\mathcal{G}_t] \\
= \; & \mathrm{E}\left[\left(\mathbb{1}_{\{\tau\le t\}} + \mathbb{1}_{\{t<\tau\le s\}}\right)\frac{1}{B_T}\left(Z_\tau - \mu^Z(\tau)\right)M_s|\mathcal{G}_t\right] \\
= \; & L_t \cdot \mathrm{E}[M_s|F_t] \\
& + \int_t^s \frac{1}{B_T}\left(\mu^Z(u) - \mu^Z(u)\right)\exp\left\{-\int_t^u \widehat{\lambda}(v)dv\right\}\widehat{\lambda}(u)\,\mathrm{E}[M_s|\mathcal{G}_t]\, du \\
= \; & L_t M_t
\end{aligned}$$

for any $s \ge t$. □

Proof of Lemma 2. From equation (23) in Bielecki et al. (2008), it follows that the discounted cumulative value of the credit derivative (Z, C, F) follows the dynamics

$$dV_t^F = (1 - H_t)\cdot\exp\left\{\int_0^t \widehat{\lambda}(s)ds\right\}\; dm_t + \left(\frac{Z(t)}{B_T} - (g_t^C + g_t^F + g_t^Z)\right)d\widetilde{H}_t,$$

hence

$$\begin{aligned}
& V_T^F \\
= \;& F_0 + \int_0^T (1-H_t)\cdot \exp\left\{\int_0^t \widehat{\lambda}(s)\, ds\right\}\, dm_t \\
& \qquad + \int_0^T \left(\frac{Z(t)}{\widehat{E}[B_T|\mathcal{F}_t]} - (g_t^C + g_t^F + g_t^Z)\right)\, d\widetilde{H}_t \\
= \;& F_0 + \int_0^T (1-H_t)\cdot \exp\left\{\int_0^t \widehat{\lambda}(s)\, ds\right\}\cdot \xi_t\, d\widehat{W}_t + \int_0^T \frac{Z(t)-\mu^Z(t)}{\widehat{E}[B_T|\mathcal{F}_t]}\, d\widetilde{H}_t \\
& \qquad + \int_0^T \left(\frac{\mu^Z(t)}{\widehat{E}[B_T|\mathcal{F}_t]} - (g_t^C + g_t^F + g_t^Z)\right)\, d\widetilde{H}_t \\
= \;& F_0 + \int_0^T \Bigg[(1-H_t)\cdot \exp\left\{\int_0^t \widehat{\lambda}(s)\, ds\right\}\cdot \frac{\xi_t}{\sigma(t)X_{t-}} \\
& \quad + \frac{g_{t-}^C + g_{t-}^F + g_{t-}^Z}{X_{t-}} - \frac{\mu^Z(t)}{\widehat{E}[B_T|\mathcal{F}_t]X_{t-}}\Bigg]\, dX_t + \int_0^T \frac{Z(t)-\mu^Z(t)}{\widehat{E}[B_T|\mathcal{F}_t]}\, d\widetilde{H}_t,
\end{aligned}$$

is the FS-decomposition of the credit derivative in case of a non-trivial reference filtration $(\mathcal{F}_t)$. In particular, the locally risk-minimizing hedge ratio is given by

$$h_t^X = (1-H_t)\cdot \exp\left\{\int_0^t \widehat{\lambda}(s)\, ds\right\}\cdot \frac{\xi_t}{\sigma(t)X_{t-}} + \frac{g_{t-}^C + g_{t-}^F + g_{t-}^Z}{X_{t-}} - \frac{\mu^Z(t)}{\widehat{E}[B_T|\mathcal{F}_t]X_{t-}}.$$

□

Proof of Lemma 3.

$$\begin{aligned}
V_T^F \;=\;& F_0 + \int_0^T (1-H_t)G_t^{-1}\, dm_t + \int_0^T \left(\frac{Z(t)}{\widehat{E}[B_T|\mathcal{F}_t]} - (g_t^C + g_t^F + g_t^Z)\right) d\widetilde{H}_t \\
=\;& F_0 + \int_0^T (1-H_t)\, \frac{1}{B_tG_t} B_t\, \frac{\xi_t^{m,r} + \xi_t^{m,\widehat{\lambda}}}{\xi_t^{X,r} + \xi_t^{X,\widehat{\lambda}}}\, dm_t^X + \int_0^T \frac{Z(t)-\mu^Z(t)}{\widehat{E}[B_T|\mathcal{F}_t]}\, d\widetilde{H}_t \\
& \quad + \int_0^T \left(\frac{\mu^Z(t)}{\widehat{E}[B_T|\mathcal{F}_t]} - (g_t^C + g_t^F + g_t^Z)\right) d\widetilde{H}_t \\
=\;& F_0 + \int_0^T \left[(1-H_t)\, B_t\, \frac{\xi_t^m}{\xi_t^X} + \frac{g_{t-}^C + g_{t-}^F + g_{t-}^Z}{X_{t-}} - \frac{\mu^Z(t)}{\widehat{E}[B_T|\mathcal{F}_t]X_{t-}}\right] dX_t \\
& \quad + \int_0^T \frac{Z(t)-\mu^Z(t)}{\widehat{E}[B_T|\mathcal{F}_t]}\, d\widetilde{H}_t.
\end{aligned}$$

□

The crucial step in deriving explicit hedge ratios in continuous-time models usually is to calculate the predictable process appearing in the martingale representation of some payoff. In our setting, we are interested in the process m with martingale representation 11. Bruti-Liberati and Platen (2010, p. 591ff.) considered the problem of finding explicit integral representations of general derivatives' payoff structures. For the reader's convenience, we state these results, which are basically due to Heath (1995), in the two propositions below.

We first consider a market driven by a single state variable, a stochastic process Y with dynamics[16]

$$dY_t = \alpha(t,Y_t)\;dt + \sigma(t,Y_t)\;dW_t.$$

Consider a European contingent claim Z whose payoff at maturity T depends on the evolution of the state variable, i.e.

$$Z = Z(\overline{Y}_T),$$

where $\overline{Y}_t = \{Y_s : s \leq t\}$ for all t. In particular, we have $\mathcal{F}_t = \sigma(W_s : s \leq t) = \sigma(\overline{Y}_t)$ and, for an $(\mathcal{F}_t)$-martingale m, the martingale representation writes

$$m_t = m_0 + \int_0^t \xi_s^m dW_s. \tag{A5}$$

We then have the following result which follows from Bruti-Liberati and Platen (2010, p. 597).

Proposition A.1 (Explicit Hedge Ratio)
Define the martingale m by $m_t = E[Z|\mathcal{F}_t]$ for all t. Suppose there exists a deterministic function $u : [0,T] \times \mathbb{R} \to \mathbb{R}$ of class[17] $C^{1,3}$ such that

$$u(t,y) = E[Z|\mathcal{F}_t]$$

for any t and y. Then, the process ξ^m in (A5) is given by

$$\xi_s^m = \sigma(s,Y_s) \cdot \frac{\partial}{\partial y} u(s,Y_s).$$

[16] In our applications, this corresponds to the case of either the interest rate or the default rate being stochastic. In this case, we have $Y_t = r_t$ respectively $Y_t = \widehat{\lambda}_t$ for all t.

[17] A function $u : [0,T] \times \mathbb{R} \to \mathbb{R}, (t,y) \mapsto u(t,y)$ is of class $C^{1,3}$, if u is continuously differentiable with respect to t and three times continuously differentiable with respect to y.

We now consider a market driven by two state variables, i.e. a two-dimensional stochastic process $Y = (Y^1, Y^2)$ with dynamics[18]

$$dY_t^i = \alpha^i(t, Y_t)\, dt + \sum_{j=1}^{2} \sigma^{i,j}(t, Y_t)\, dW_t^i. \tag{A6}$$

for $i = 1,2$. Consider a European contingent claim Z whose payoff at maturity T depends on the evolution of the two state variables, i.e.

$$Z = Z(\overline{Y}_T^1, \overline{Y}_T^2),$$

where $\overline{Y}_t^i = \{Y_s^i : s \leq t\}$ for all t, $i = 1,2$. In this case, the martingale representation writes

$$m_t = m_0 + \int_0^t \xi_s^{m,1} dW_s^1 + \int_0^t \xi_s^{m,2} dW_s^2. \tag{A7}$$

We now state the explicit formula for the processes $\xi_s^{m,i}$, $i = 1,2$, in case the state variable Y^i only depends on W^i, $i = 1,2$. In particular, we then have

$$\sigma^{i,j} = \delta^{i,j} \cdot \sigma^{i,i} \tag{A8}$$

in (A6), where δ denotes the Kronecker delta. The following result is a direct consequence from Bruti-Liberati and Platen (2010, p. 605).

Proposition A.2 (Explicit Hedge Ratio, Case of a Two-Dimensional BM) *Define the martingale m by $m_t = E[Z|\mathcal{F}_t]$ for all t. Suppose there exists a deterministic function $u : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ of class $C^{1,3}$ such that*

$$u(t, y^1, y^2) = E[Z|\mathcal{F}_t]$$

for any t and y. Then, the processes $\xi^{m,i}$, $i = 1,2$, in (A7) are given by

$$\begin{aligned} \xi_s^{m,1} &= \sigma^{1,1}(s, Y_s) \cdot \frac{\partial}{\partial y^1} u(s, Y_s), \\ \xi_s^{m,2} &= \sigma^{2,2}(s, Y_s) \cdot \frac{\partial}{\partial y^2} u(s, Y_s). \end{aligned}$$

[18] In our applications, this corresponds to the case of both the interest rate and the default rate being stochastic. In this case, we have $Y_t^1 = r_t$ and $Y_t^2 = \widehat{\lambda}_t$ for all t.

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